Level one automorphic representations of an anisotropic exceptional group over $\mathbb{Q}$ of type F_4

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Abstract

Up to isomorphism, there is a unique connected semisimple algebraic group over $\mathbb{Q}$ of Lie type F_4, with compact real points and split over $\mathbb{Q}_p$ for all primes p. Let $\mathbf{F}_4$ be such a group. In this paper, we study the level one automorphic representations of $\mathbf{F}_4$ in the spirit of the work of Chenevier, Renard and Taïbi [CR15; Taï17; CT20].

First, we give an explicit formula for the number of these representations having any given archimedean component. For this, we study the automorphism group of the two definite exceptional Jordan algebras of rank 27 over $\mathbb{Z}$ studied by Gross in [Gro96], as well as the dimension of the invariants of these groups in all irreducible representations of $\mathbf{F}_4(\mathbb{R})$.

Then, assuming standard conjectures by Arthur and Langlands for $\mathbf{F}_4$ [Art89; CL19], we refine this counting by studying the contribution of the representations whose global Arthur parameter has any possible image (or "Sato-Tate group"). This includes a detailed description of all those images, as well as precise statements for the Arthur's multiplicity formula in each case. As a consequence, we obtain a conjectural but explicit formula for the number of algebraic, cuspidal, level one automorphic representation of GL_{26} over $\mathbb{Q}$ with Sato-Tate group $\mathbf{F}_4(\mathbb{R})$ of any given weight (assumed "F_4-regular"). The first example of such representations occurs in motivic weight 36.

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1 Introduction

1.1 Galois representations with given image

The absolute Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ encodes a lot of arithmetic information about number fields, and a natural way to study $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ is to consider its representations, especially those arising from algebraic geometry. Motivated by the inverse Galois problem, the following question has been studied by a lot of mathematicians:

Question 1. *Let ℓ be a prime number and H a connected reductive algebraic group over $\overline{\mathbb{Q}_\ell}$. Is there an ℓ-adic Galois representation $\rho : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to H(\overline{\mathbb{Q}_\ell})$ such that it is semisimple and geometric (in the sense of Fontaine-Mazur [Tay04, Conjecture 1.1]), and its image is Zariski dense in $H(\overline{\mathbb{Q}_\ell})$?*

In the case $H = \mathrm{GL}_2 \simeq \mathrm{GSp}_2$ or GSp_4, or more generally, a (similitude) classical group, there are many well-known constructions and examples. For instance, one can use the Poincaré pairing on ℓ-adic cohomologies of algebraic varieties to construct Galois representations with images in classical groups. The case of exceptional groups, i.e. groups with Lie types G_2, F_4, E_6, E_7 and E_8, is harder, but we still have some examples in [DR10; GS98; Yun14; Pat16; BCELMP19]. Notice that when H has Lie type G_2 or E_8, this question is related to Serre's question on motives [Ser94, Question 8.8, §1].

Composing $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to H(\overline{\mathbb{Q}_\ell})$ with an irreducible faithful algebraic representation $H \hookrightarrow \mathrm{GL}_n$, we obtain an n-dimensional geometric ℓ-adic representation. One can associate two invariants with a geometric ℓ-adic Galois representation $\rho : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_n(\overline{\mathbb{Q}_\ell})$: the *(Artin) conductor* $\mathrm{N}(\rho) \in \mathbb{N}$, and the *Hodge-Tate weights* $\mathrm{HT}(\rho)$, a multiset of n integers (see, for example, [Tay04]). In the aforementioned works, the conductors of the geometric ℓ-adic representations that they construct are usually not controlled. One may refine Question 1 naturally by fixing these two invariants:

Question 2. *Let ℓ be a prime number, $n \geq 1$ and H a connected reductive subgroup of GL_n over $\overline{\mathbb{Q}_\ell}$. What is the number (up to equivalence) of geometric ℓ-adic Galois representations $\rho : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_n(\overline{\mathbb{Q}_\ell})$ of given conductor and Hodge-Tate weights such that the Zariski closure of $\mathrm{Im}(\rho)$ is $H(\overline{\mathbb{Q}_\ell})$?*

For $(H, n) = (\mathrm{GL}_2, 2)$ or $(\mathrm{SO}_{2g+1}, 2g+1)$, this question is for instance related to the dimension of spaces of classical or Siegel modular forms. We have less knowledge of the cases of other groups H. When the conductor $N = 1$, Question 2 is solved *conjecturally* by Chenevier and Renard in [CR15] for the following groups (n is chosen to be the dimension of the standard representation when H is a (similitude) classical group, and to be 7 when H has type G_2):

$$\mathrm{GL}_2 \simeq \mathrm{GSp}_2,\ \mathrm{GSp}_4,\ \mathrm{SO}_4,\ \mathrm{SO}_5,\ \mathrm{GSp}_6,\ \mathrm{GSp}_8,\ \mathrm{SO}_8,\ \mathrm{G}_2,$$

via the conjectural connection between n-dimensional geometric ℓ-adic representations and cuspidal automorphic representations of GL_n. See also [Taï17; CT20] for higher dimensions.

In [Lac20], Lachaussée extends the results for GSp_{2g}, $1 \leq g \leq 4$ to the case of Artin conductor $N = 2$. Now we concentrate on the case of conductor one (see Remark 1.6.4 for more explanations about this conductor one assumption).

In this paper, following [CR15], we give a *conjectural* solution to Question 2 when $N = 1$, H has Lie type F_4, and $n = 26$. For a 26-dimensional geometric ℓ-adic Galois representation ρ such that $\overline{\mathrm{Im}(\rho)}$ has type F_4, its multiset of Hodge-Tate weights only depends on 4 variables $a, b, c, d \in \mathbb{N}$, and has the form

$$\mathrm{HT}(a,b,c,d) := \left\{ \begin{array}{c} 0, 0, \pm a, \pm b, \pm(a+b), \pm(b+c), \pm(a+b+c), \pm(b+c+d), \pm(a+b+c+d), \pm(a+2b+c), \\ \pm(a+2b+c+d), \pm(a+2b+2c+d), \pm(a+3b+2c+d), \pm(2a+3b+2c+d). \end{array} \right\}$$

As a conjectural corollary of our results in this paper, we propose the following conjecture on F_4-type geometric ℓ-adic representations:

Conjecture A. *The number of equivalence classes of* 26*-dimensional conductor one geometric ℓ-adic Galois representations ρ such that*

- *the Zariski closure of* $\mathrm{Im}(\rho)$ *is a connected reductive group of type* F_4*,*
- *and* $\mathrm{HT}(\rho) = \mathrm{HT}(a,b,c,d)$*,* $a, b, c, d \geq 1$*,*

is $\mathrm{F}_4(a-1, b-1, c-1, d-1)$*, where* $\mathrm{F}_4(\lambda)$ *is the computable function on* $\mathbb{N}^4$ *given by Proposition 6.4.1.*

Remark 1.1.1. The formula for $\mathrm{F}_4(\lambda)$ has so many terms that we will not write down the full formula in this paper. However, under some hypothesis this formula becomes much simpler. For instance, when $a > b + c + d + 3$, $b, c, d > 0$ and c, d are both odd, a short formula for $\mathrm{F}_4(a, b, c, d)$ is given in Remark 6.4.2.

Example 1.1.2. Among quadruples (a, b, c, d) with nonzero $\mathrm{F}_4(a, b, c, d)$, there exists a unique one $(1, 2, 0, 2)$ that has the smallest $2a+3b+2c+d$ (the largest Hodge-Tate weight). Moreover, $\mathrm{F}_4(1, 2, 0, 2) = 1$, so according to Conjecture A there should be a unique 26-dimensional conductor one geometric ℓ-adic representation ρ such that

- $\overline{\mathrm{Im}(\rho)}$ has type F_4,
- and its multiset of Hodge-Tate weights $\mathrm{HT}(\rho)$ is:

$$\mathrm{HT}(2, 3, 1, 3) = \{0, 0, \pm 2, \pm 3, \pm 4, \pm 5, \pm 6, \pm 7, \pm 9, \pm 9, \pm 12, \pm 13, \pm 16, \pm 18\}.$$

For people preferring non-negative Hodge-Tate weights, one can twist ρ by ω_ℓ^{-18}, where ω_ℓ denotes the ℓ-adic cyclotomic character of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, and obtain a representation with motivic weight 36. Hence we expect a 26-dimensional geometric ℓ-adic representation whose Zariski image is the product of an F_4-type group with $\overline{\mathbb{Q}_\ell}^\times$ to appear in the 36th degree ℓ-adic cohomology of some algebraic variety. A very interesting open problem is to find such a variety!

1.2 An automorphic variant of Question 2

Now we send Question 2 to the automorphic side. Let G be a connected reductive group over $\mathbb{Q}$ with a reductive $\mathbb{Z}$-model (see §2.2). As we will talk about Galois representations, it will be convenient to assume that $\widehat{G}$ is defined over $\overline{\mathbb{Q}}$, and we fix two embeddings: $\iota_\infty : \overline{\mathbb{Q}} \to \mathbb{C}$ and $\iota_\ell : \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}_\ell}$. We also fix a maximal compact subgroup G_c of $\widehat{G}(\mathbb{C})$.

Let π be an L-*algebraic* [1] level one automorphic representation of G. By a conjecture of Buzzard and Gee [BG14, Conjecture 3.2.1], one should be able to associate with π a compatible conductor one geometric ℓ-adic representation $\rho_{\pi,\iota} : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \widehat{G}(\overline{\mathbb{Q}_\ell})$, which depends on the choice of embeddings $\iota = (\iota_\infty, \iota_\ell)$. By the standard conjectures of Fontaine-Mazur and Langlands, every conductor one geometric ℓ-adic representation into $\widehat{G}(\overline{\mathbb{Q}_\ell})$ should arise in this way. If any two element-conjugate homomorphisms from a connected compact Lie group into G_c are conjugate (see §4.1 for a detailed explanation), the following question gives an automorphic variant of Question 2 for $H = \widehat{G} \times_{\iota_\ell} \overline{\mathbb{Q}_\ell}$:

Question 3. *Let G be a connected reductive group over $\mathbb{Q}$ admitting a reductive $\mathbb{Z}$-model.*

*(1) (*Counting*) Count the number (up to equivalence) of level one algebraic* [2] *discrete automorphic representations for G with an arbitrary given archimedean component.*

*(2) (*Refinement*) Refine this counting by* “Sato-Tate groups” *of automorphic representations.*

Remark 1.2.1 (“Sato-Tate groups”). In the above question, the “Sato-Tate group” $\mathrm{H}(\pi)$ of a level one automorphic representation π for G is a certain conjugacy class of subgroups of G_c that we will explain carefully in §5.3.1, and we can briefly introduce it as follows. Based on Arthur’s parametrization of automorphic representations, one can *conjecturally* associate with π a group homomorphism

$$\psi_\pi : \mathcal{L}_{\mathbb{Z}} \times \mathrm{SU}(2) \to G_c,$$

where $\mathcal{L}_{\mathbb{Z}}$ is the *hypothetical Langlands group*, which is connected and compact (see §5.3). We define $\mathrm{H}(\pi)$ to be the conjugacy class of the image of ψ_π in G_c. When the restriction of ψ_π to $1 \times \mathrm{SU}(2) \subset \mathcal{L}_{\mathbb{Z}} \times \mathrm{SU}(2)$ is trivial, this notion $\mathrm{H}(\pi)$ coincides with the usual notion of Sato-Tate groups. In general, we decided to include the $\mathrm{SU}(2)$ factor in the definition as it provides convenience for stating some of our results.

The point of the refinement part in Question 3 is that in general many level one discrete automorphic representations π for G, for example the *endoscopic* ones, will have a Sato-Tate group strictly smaller than G_c. For these π, $\overline{\mathrm{Im}(\rho_{\pi,\iota})}$ should be a proper subgroup of $\widehat{G}(\overline{\mathbb{Q}_\ell})$. Hence we have to find a way to exclude these representations to obtain the desired number in Question 2.

In [CR15], Chenevier and Renard solve the part (1) of Question 3 for a number of classical groups of small ranks, namely, G is one of the following groups:

$$\mathrm{SL}_2 = \mathrm{Sp}_2,\ \mathrm{Sp}_4,\ \mathrm{SO}_{2,2},\ \mathrm{SO}_{3,2},\ \mathrm{SO}_7,\ \mathrm{SO}_8\ \text{ and }\ \mathrm{SO}_9,$$

and also for a connected semisimple $\mathbb{Q}$-group of type G_2 with compact real points. For the part (2) of Question 3, their method relies in an important way on Arthur’s classification of automorphic representations [Art89; Art13]. Their results for $\mathrm{SO}_7, \mathrm{SO}_8, \mathrm{SO}_9$ and G_2 are conditional to Arthur’s conjectures for these groups, since $\mathrm{SO}_7, \mathrm{SO}_8$ and SO_9 are not quasi-split, and G_2 is not covered by Arthur’s results. In [Taï17], Taïbi uses Arthur’s L^2-Lefschetz

[1] For the definition of L-algebraicity, see [BG14, Definition 2.3.1]. For a representation which is algebraic in the sense of Definition 5.4.3 but not L-algebraic, one should replace $\widehat{G}$ by some “similitude” group.

[2] One can remove this algebraicity condition by restricting to semisimple $\mathbb{Q}$-groups.

formula to make these results unconditional (except for G_2) and he also extends them to the following split classical groups:

$$\mathrm{Sp}_{2g} \text{ with } g \leq 7,\ \mathrm{SO}_{n+1,n} \text{ with } n \leq 8 \text{ and } \mathrm{SO}_{2m,2m} \text{ with } m \leq 4.$$

In particular, Taïbi's solution to Question 3 for Sp_8 will be important in our work.

In this paper, we apply the method of [CR15] to $\mathbf{F}_4$, the unique (up to isomorphism) connected semisimple algebraic group over $\mathbb{Q}$ of type F_4, with compact real points and split over $\mathbb{Q}_p$ for every prime p. The construction of $\mathbf{F}_4$ is explicitly given in §2.1. For this group, automorphic representations are automatically L-algebraic. Moreover, it turns our that there is no local-global conjugacy problems for connected subgroups of $(\mathbf{F}_4)_c = \mathbf{F}_4(\mathbb{R})$ (see Proposition 4.1.5). As a consequence, Conjecture A follows from standard conjectures and our result on automorphic representations (Theorem F).

Remark 1.2.2. The automorphic representations for $\mathbf{F}_4$ (and their local components) have been studied in [Sav94; MS97; Gan00; Pol23; KS23] via exceptional theta correspondences, and we will explain some links between these correspondences with our work in §6.5. Let us mention also that automorphic representations for $\mathbf{F}_4$ have also been studied in the past by Seth Padowitz in [Pad98, §9]. Padowitz rather considers the automorphic representations which are Steinberg at a fixed *non-empty* set of primes and unramified elsewhere, and tries to enumerate them using the stable trace formula, in the spirit of works of Gross-Pollack [GP05]. The results are only partial, as several stable local orbital integrals there are not determined [3], and we hope to go back to this question in the future.

1.3 Counting level one automorphic representations

In [Gro96], Gross proves the following result for $\mathbf{F}_4$, which is important in our solution to the part (1) of Question 3 for $\mathbf{F}_4$:

Theorem B. *(Proposition 2.3.5) Up to $\mathbb{Z}$-isomorphism, there are two smooth affine group schemes over $\mathbb{Z}$ with generic fiber isomorphic to $\mathbf{F}_4$, whose special fiber over $\mathbb{Z}/p\mathbb{Z}$ is reductive for all primes p.*

The $\mathbb{Z}$-group schemes in Theorem B are reductive $\mathbb{Z}$-models of $\mathbf{F}_4$. Their constructions are related to integral structures of the 27-dimensional definite exceptional Jordan algebra over $\mathbb{Q}$. Gross proves this result via the mass formula for $\mathbf{F}_4$ and some results in [ATLAS], and we will give a new proof in §2.3 without using [ATLAS].

In our proof of Theorem B, we study the $\mathbb{Z}$-points of two reductive $\mathbb{Z}$-models in Theorem B, which are finite groups inside the compact Lie group $\mathbf{F}_4(\mathbb{R})$. With the help of [PARI/GP] and [GAP], for each of these finite groups, we give an explicit set of generators in §3.2 and enumerate its conjugacy classes in §3.3.

Since the method of counting in [CR15] can be applied to any algebraic $\mathbb{Q}$-group that has compact real points and admits a reductive $\mathbb{Z}$-model, we recall and apply this method to $\mathbf{F}_4$ in §3.1, §3.4 and §3.5. This formula leads to the answer for the part (1) of Question 3 in the case of $\mathbf{F}_4$, which is also the main computational result in this paper:

[3]Another minor problem is that the author asserts on [Pad98, P.42] that the 26-dimensional irreducible representation of $\mathbf{F}_4$ is "excellent" in his sense, which is not correct. See Remark 3.5.5 for a conterexample.

Theorem C. *(Theorem 3.6.1 and Corollary 5.1.8) (1) For an irreducible representation* V_λ *of* $\mathbf{F}_4(\mathbb{R})$ *with highest weight* λ, *we have an explicit and computable formula for the number* $d(\lambda)$ *of equivalence classes of level one automorphic representations* π *with* $\pi_\infty \simeq \mathrm{V}_\lambda$.

(2) For dominant weights $\lambda = \sum_{i=1}^{4} \lambda_i \varpi_i$[4] *satisfying* $2\lambda_1 + 3\lambda_2 + 2\lambda_3 + \lambda_4 \leq 13$, *we list the numbers* $d(\lambda)$ *in Table 6, Appendix A.*

1.4 Candidates for Sato-Tate groups

The part (2) of Question 3 involves a classification of all possible Sato-Tate groups for level one automorphic representations of $\mathbf{F}_4$. For this $\mathbb{Q}$-group, its Langlands dual group $\widehat{\mathbf{F}_4}$ is isomorphic to $\mathbf{F}_4 \times_{\mathbb{Q}} \mathbb{C}$, and as mentioned in Remark 1.2.1, Sato-Tate groups in this case are conjugacy classes of subgroups of the compact Lie group $\mathbf{F}_4(\mathbb{R})$. The following result gives us 13 candidates for Sato-Tate groups strictly smaller than $\mathbf{F}_4(\mathbb{R})$:

Theorem D. *(Theorem 4.6.7) There are* 13 *conjugacy classes of proper connected subgroups* H *of* $\mathbf{F}_4(\mathbb{R})$ *such that:*

- *the centralizer of* H *in* $\mathbf{F}_4(\mathbb{R})$ *is isomorphic to the product of finitely many copies of* $\mathbb{Z}/2\mathbb{Z}$*;*
- *the zero weight appears twice in the restriction of the* 26*-dimensional irreducible representation of* $\mathbf{F}_4(\mathbb{R})$ *to* H*.*

We prove this classification result step by step in §4.3, §4.4, §4.5 and §4.6, following Dynkin's strategy in [Dyn57]. It is worth mentioning two important ingredients in the proof:

- A local-global conjugacy result (Proposition 4.1.5) for $\mathbf{F}_4(\mathbb{R})$, which we have already mentioned in the end of §1.2. This relies on a result about Lie algebras (Theorem 4.1.3) proved by Losev in [Los10].
- A useful criterion (Proposition 4.2.1) given in §4.2 for the conjugacy of two homomorphisms from a connected compact Lie group into $\mathbf{F}_4(\mathbb{R})$.

Example 1.4.1. Among the conjugacy classes of subgroups classified in Theorem D, we have

$$\mathrm{Spin}(9),\ \mathrm{Spin}(8),\ \mathrm{G}_2 \times \mathrm{SO}(3),\ \left(\mathrm{Sp}(3) \times \mathrm{SU}(2)\right)/\mu_2^\Delta,\ \left(\mathrm{Sp}(2) \times \mathrm{SU}(2) \times \mathrm{SU}(2)\right)/\mu_2^\Delta,$$

where the notations will be explained in Notation 4.3.1 and Notation 4.3.3. The remaining subgroups are all centrally isogenous to products of n copies of $\mathrm{SU}(2)$, $n \leq 4$. Note that among the subgroups listed above, only $\mathrm{Spin}(9)$ and $\left(\mathrm{Sp}(3) \times \mathrm{SU}(2)\right)/\mu_2^\Delta$ are maximal proper connected regular subgroups of $\mathbf{F}_4(\mathbb{R})$.

1.5 Arthur's conjectures

As in [CR15], for the part (2) of Question 3, we need some conjectures on automorphic representations. For a connected reductive algebraic group G over $\mathbb{Q}$, Arthur introduces in [Art89] a conjectural parametrization of discrete automorphic representations, via *discrete*

[4] Here we follow the notations in [Bou07, §IV.4.9].

global Arthur parameters for G. In the level one case, these parameters are $\widehat{G}(\mathbb{C})$-conjugacy classes of admissible morphisms

$$\psi : \mathcal{L}_{\mathbb{Z}} \times \mathrm{SL}_2(\mathbb{C}) \to \widehat{G}(\mathbb{C}),$$

where $\mathcal{L}_{\mathbb{Z}}$ is the hypothetical Langlands group of $\mathbb{Z}$ (see §5.3 for more details), and $\widehat{G}$ is the Langlands dual group of G. Arthur proposes a conjectural formula for the multiplicity of an irreducible $G(\mathbb{A})$-representation in the discrete automorphic spectrum of G, in terms of the associated global Arthur parameters.

In [Art13], Arthur reformulates his conjectures for any quasi-split classical group G, avoiding the appearance of the hypothetical Langlands group $\mathcal{L}_{\mathbb{Z}}$. In this case, he relates the global Arthur parameters for G to cuspidal automorphic representations of linear groups, and proves the endoscopic classifications, relying in particular on the works of Mœglin-Waldspurger [MW14], Ngô [Ngô10] and many others. We refer to [CL19, §8] for precise statements of Arthur's results in [Art13] in the case of level one cohomological automorphic representations of classical groups.

Of course $\mathbf{F}_4$ is not a classical group, and Arthur's general conjectures [Art89] are still open in this case. Nevertheless, they can still be formulated quite precisely if we admit the existence of $\mathcal{L}_{\mathbb{Z}}$. See also [CL19, §6.4] for some generalities of Arthur's conjectures in the level one case.

Notation 1.5.1. In the rest of this paper, we will mark any result conditional to the existence of $\mathcal{L}_{\mathbb{Z}}$ and Arthur's multiplicity formula (Conjecture 5.6.5) with a star $*$.

Now we briefly explain Arthur's conjectures for $\mathbf{F}_4$. For a level one automorphic representation π of $\mathbf{F}_4$ with global Arthur parameter $\psi : \mathcal{L}_{\mathbb{Z}} \times \mathrm{SL}_2(\mathbb{C}) \to \mathbf{F}_4(\mathbb{C})$, we may compose ψ with the 26-dimensional irreducible representation $\mathrm{r} : \mathbf{F}_4(\mathbb{C}) \to \mathrm{GL}_{26}(\mathbb{C})$ [5], and thus obtain a representation of $\mathcal{L}_{\mathbb{Z}} \times \mathrm{SL}_2(\mathbb{C})$. This representation is decomposed as:

$$\mathrm{r} \circ \psi \simeq \pi_1[d_1] \oplus \cdots \oplus \pi_k[d_k], \tag{$\star$}$$

where π_i is an n_i-dimensional irreducible representation of $\mathcal{L}_{\mathbb{Z}}$ and $[d_i]$ stands for the irreducible d_i-dimensional representation of $\mathrm{SL}_2(\mathbb{C})$, and $\sum_{i=1}^{k} n_i d_i = 26$. We identify π_i as a level one cuspidal representations of PGL_{n_i}, and observe that it is always self-dual and algebraic in this case (see §5.4). In a similar way as in [Art13], we view the global Arthur parameter ψ as a linear combination of $\pi_i[d_i]$'s.

In §6.1, we derive from Theorem D that the Sato-Tate group of any π_i appearing in the decomposition ($\star$) is one of the following compact Lie groups:

$$\mathrm{SU}(2),\ \mathrm{Sp}(2),\ \mathrm{Sp}(3),\ \mathrm{SO}(8),\ \mathrm{SO}(9),\ \mathrm{G}_2, \mathbf{F}_4(\mathbb{R}). \tag{$\star\star$}$$

Cuspidal representations with Sato-Tate group $\mathbf{F}_4(\mathbb{R})$ conjecturally correspond to the desired ℓ-adic representations in Question 2, and those with other Sato-Tate groups in ($\star\star$) are related to level one automorphic representations for the following $\mathbb{Q}$-groups:

$$\mathrm{PGL}_2,\ \mathrm{SO}_{3,2},\ \mathrm{SO}_7,\ \mathrm{SO}_8,\ \mathrm{Sp}_8, \mathrm{G}_2,$$

[5]The image of r is even inside $\mathrm{SO}_{26}(\mathbb{C}) \subset \mathrm{SL}_{26}(\mathbb{C}) \subset \mathrm{GL}_{26}(\mathbb{C})$.

which have already been studied in [CR15; Taï17; CT20].

Conversely, for a global Arthur parameter $\psi : \mathcal{L}_{\mathbb{Z}} \times \mathrm{SL}_2(\mathbb{C}) \to \mathbf{F}_4(\mathbb{C})$ whose "archimedean component" is an Adams-Johnson parameter (see Definition 5.6.1 and Remark 5.6.2), the multiplicity of its corresponding irreducible $\mathbf{F}_4(\mathbb{A})$-representation π in the automorphic spectrum can be calculated via Arthur's formula in [Art89]. In the level one case, this formula involves two characters on the centralizer C_ψ of $\mathrm{Im}(\psi)$ in $\mathbf{F}_4(\mathbb{C})$, which is an elementary abelian 2-group. The first character is Arthur's character ε_ψ, and we will recall its definition in §5.6.2. The second character is a local character at the archimedean place, an explicit formula for which will be given in §6.2.

1.6 Refinement of the counting

With all these preparations, we are ready to refine the counting in Theorem C. For a global Arthur parameter $\psi : \mathcal{L}_{\mathbb{Z}} \times \mathrm{SL}_2(\mathbb{C}) \to \mathbf{F}_4(\mathbb{C})$, one can associate two invariants:

- its Sato-Tate group $\mathrm{H}(\psi) := \psi(\mathcal{L}_{\mathbb{Z}} \times \mathrm{SU}(2))$, viewed as a conjugacy of subgroups in the compact group $\mathbf{F}_4(\mathbb{R})$;
- its "weights", i.e. eigenvalues of its infinitesimal character under the 26-dimensional irreducible representation $\mathrm{r} : \mathbf{F}_4 \to \mathrm{SL}_{26}$.

Given any conjugacy class of proper subgroups H of $\mathbf{F}_4(\mathbb{R})$ classified in Theorem D, in §6.3 we classify all the possible decompositions ($\star$) of $\mathrm{r} \circ \psi$ for global Arthur parameters ψ with $\mathrm{H}(\psi) = H$. If ψ corresponds to an irreducible level one $\mathbf{F}_4(\mathbb{A})$-representation π, an important part of our work is to give an exact formula for the multiplicity of π, for each case of Sato-Tate groups. Roughly speaking, the multiplicity depends on how the weights of ψ are distributed in the summands $\pi_i[d_i]$'s of ($\star$). In conclusion, we have the following result:

Theorem* E. *(Theorem 6.3.1)*

(a) The Sato-Tate group of a level one automorphic representation for $\mathbf{F}_4$ is either $\mathbf{F}_4(\mathbb{R})$ or one of the proper subgroups of $\mathbf{F}_4(\mathbb{R})$ classified in Theorem D except $\mathrm{Spin}(8)$*.*

*(b) For global Arthur parameters of $\mathbf{F}_4$ with a given Sato-Tate group, the multiplicity of its corresponding irreducible level one $\mathbf{F}_4(\mathbb{A})$-representation (*0 *or* 1*) is given explicitly by the formulas in Proposition 6.3.4 to Proposition 6.3.18.*

Remark 1.6.1. We observe that not all subgroups in Theorem D come from *endoscopic groups* of $\mathbf{F}_4$, in the sense of [Art13]. For example, the subgroup $\mathrm{G}_2 \times \mathrm{SO}(3)$ has trivial centralizer in $\mathbf{F}_4(\mathbb{R})$, thus it can not be the centralizer of any element in $\mathbf{F}_4(\mathbb{R})$. As a result, our conjectural refinement is finer than Arthur's endoscopic classification in [Art13].

Given an irreducible representation V_λ of $\mathbf{F}_4(\mathbb{R})$, from Theorem C we know the number of equivalence classes of level one automorphic representations π for $\mathbf{F}_4$ with $\pi_\infty \simeq \mathrm{V}_\lambda$. The weights of the global Arthur parameter ψ_π of π are determined by V_λ. We can enumerate all the possible global Arthur parameters with these weights, and then use the multiplicity formulas in Theorem E to determine their multiplicities. In this way, we obtain a *conjectural* refinement of the counting in Theorem C.

Example 1.6.2. In Table 9 and Table 10, we list some parameters with "small" archimedean components. For example, there are two different level one automorphic representations of

$\mathbf{F}_4$ with trivial archimedean components, whose Arthur parameters are:

$$[9] \oplus [17] \text{ and } \Delta_{11}[6] \oplus [5] \oplus [9].$$

The first parameter corresponds to the trivial representation, and its Sato-Tate group [6] is the principal $\mathrm{SU}(2)$ in $\mathbf{F}_4(\mathbb{R})$. The Sato-Tate group of the second parameter is isomorphic to $(\mathrm{SU}(2) \times \mathrm{SU}(2)) / \mu_2^{\Delta}$ [7], the information about which can be found in §6.3.2. The Hecke eigenvalues of its corresponding automorphic representation for $\mathbf{F}_4$ are thus related to the Fourier coefficients of Ramanujan's Δ function, i.e. the unique level one classical cuspidal modular form with weight 11.

As a consequence of Theorem E, we obtain a conjectural solution to Question 2, stated in terms of automorphic representations:

Theorem* F. *(Proposition 6.4.1 and Proposition 6.4.3) The number of algebraic, cuspidal, level one automorphic representations of* GL_{26} *over* $\mathbb{Q}$ *satisfying:*

- *the Sato-Tate group is* $\mathbf{F}_4(\mathbb{R})$*,*
- *and the multiset of weights* [8] *is* $\mathrm{HT}(a,b,c,d)$ *for* $a,b,c,d \geq 1$*,*

is $\mathrm{F}_4(a-1,b-1,c-1,d-1)$*, where* $\mathrm{F}_4(\lambda)$ *is an explicit function on* $\mathbb{N}^4$ *given by Proposition 6.4.1.*

Example 1.6.3. The quadruples $(a,b,c,d) \in \mathbb{N}^4$ such that

- the largest weight $2a+3b+2c+d+8$ in the multiset $\mathrm{HT}(a+1,b+1,c+1,d+1)$ is not larger than 22,
- and $\mathrm{F}_4(a,b,c,d) \neq 0$,

are listed in Table 11, Appendix A. We also list the values of $\mathrm{F}_4(a,b,c,d)$ for these quadruples. As a direct consequence, we predict the existence of the geometric ℓ-adic representation in Example 1.1.2.

Remark 1.6.4. One may want to remove the level one condition, like in [Lac20]. For the part (1) of Question 3 for $\mathbf{F}_4$, one can calculate the dimension of invariants under other congruence subgroups, and obtain results similar to Theorem C for higher levels. However, for the part (2) of Question 3 for $\mathbf{F}_4$, what we use is a simplified version of Arthur's recipe in [Art89]. When allowing ramifications at some finite place p, one needs some properties of *local Arthur packets* for $\mathbf{F}_4(\mathbb{Q}_p)$, which are still unknown to us.

Let us end the introduction with a short summary of the contents of this paper. In §2, we recall the definition of $\mathbf{F}_4$ and some results of Gross [Gro96] on reductive $\mathbb{Z}$-models of $\mathbf{F}_4$. We also give a new proof for Theorem B. We prove Theorem C in §3. In §4, we study the subgroups of the compact Lie group $\mathbf{F}_4(\mathbb{R})$ and prove Theorem D. In §5, we recall the theory of level one automorphic representations and the conjectures by Arthur

[6]As we mentioned in Remark 1.2.1, the notion of Sato-Tate groups in the introduction coincides with the usual notion if and only if the restriction of the global Arthur parameter to $\mathrm{SL}_2(\mathbb{C})$ is trivial. Here these two Arthur parameters fail to satisfy this condition.

[7]Beware that there are many distinct conjugacy classes of subgroups of $\mathbf{F}_4(\mathbb{R})$ isomorphic to $\mathrm{SU}(2)$.

[8]See §5.4 for the precise definition of weights for an algebraic cuspidal level one automorphic representation of GL_n.

and Langlands, mainly following [CR15; CL19]. Then we apply these conjectures to $\mathbf{F}_4$ and prove Theorem E and Theorem F in §6.3. In Appendix A, some figures and tables used in this article are provided.

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2 The compact Lie group F_4 and its reductive integral models

In this section we introduce the compact Lie group of type F_4 that we will discuss in this paper, and give a classification of its reductive $\mathbb{Z}$-models.

2.1 The compact group F_4 and its rational structure

To construct Lie groups of exceptional types, we need to recall the notion of octonions, and our main reference is [Con12, §5].

Definition 2.1.1. An *octonion algebra* C over a field k is a non-associative k-algebra of k-dimension 8 with 2-sided identity element e such that there exists a non-degenerate quadratic form N on C satisfying $N(xy) = N(x)N(y), x, y \in C$. The quadratic form N is referred as the *norm* on C.

When considering octonion algebras over $\mathbb{R}$, we have the following classification result:

Proposition 2.1.2. *[Ada96, Theorem 15.1] Up to $\mathbb{R}$-algebra isomorphism, there is a unique octonion algebra $\mathbb{O}_{\mathbb{R}}$ over $\mathbb{R}$ whose norm N is positive definite, which is named as the* real octonion division algebra.

The multiplication law $\mathbb{O}_{\mathbb{R}} \times \mathbb{O}_{\mathbb{R}} \to \mathbb{O}_{\mathbb{R}}$ can be given as follows: as a vector space $\mathbb{O}_{\mathbb{R}}$ admits a basis $\{\mathrm{e}, \mathrm{e}_1, \ldots, \mathrm{e}_7\}$ such that e is the identity element and as an $\mathbb{R}$-algebra $\mathbb{O}_{\mathbb{R}}$ is generated by $\{\mathrm{e}_1, \ldots, \mathrm{e}_7\}$ subject to the relations

- for all i, $\mathrm{e}_i^2 = -\mathrm{e}$;
- considering the subscripts as elements in $\mathbb{Z}/7\mathbb{Z}$, the subspace of $\mathbb{O}_{\mathbb{R}}$ generated by $\{\mathrm{e}, \mathrm{e}_i, \mathrm{e}_{i+1}, \mathrm{e}_{i+3}\}$ is an associative algebra with relations

$$\mathrm{e}_i^2 = \mathrm{e}_{i+1}^2 = \mathrm{e}_{i+3}^2 = -\mathrm{e}, \mathrm{e}_i\mathrm{e}_{i+1} = -\mathrm{e}_{i+1}\mathrm{e}_i = \mathrm{e}_{i+3}.$$

We identify the real numbers $\mathbb{R}$ with the subalgebra $\mathbb{R}\mathrm{e}$ of $\mathbb{O}_{\mathbb{R}}$ and the identity element of $\mathbb{O}_{\mathbb{R}}$ will be denoted as 1. Now we recall some basic properties of $\mathbb{O}_{\mathbb{R}}$, for which we refer to [Con12, §5]. There is an anti-involution of algebra $x \mapsto \overline{x}$ called the *conjugation* on $\mathbb{O}_{\mathbb{R}}$, defined by $\overline{1} = 1$ and $\overline{\mathrm{e}_i} = -\mathrm{e}_i$ for each i. The *trace* and *norm* on $\mathbb{O}_{\mathbb{R}}$ are defined as:

$$\mathrm{Tr}(x) = x + \overline{x},\ \mathrm{N}(x) = x \cdot \overline{x} = \overline{x} \cdot x.$$

The multiplication law on $\mathbb{O}_{\mathbb{R}}$ implies that

$$\mathrm{Tr}(xy) = \mathrm{Tr}(yx) = \mathrm{Tr}(\overline{x} \cdot \overline{y}) \text{ for all } x, y \in \mathbb{O}_{\mathbb{R}}. \tag{2.1}$$

For an element $x = x_0 + \sum_{i=1}^{7} x_i \mathrm{e}_i \in \mathbb{O}_{\mathbb{R}}$, its norm $\mathrm{N}(x)$ equals $\sum_{i=0}^{7} x_i^2$, from which we can see that N is a positive definite quadratic form. Its associated symmetric bilinear form is $\langle x, y \rangle := \mathrm{N}(x+y) - \mathrm{N}(x) - \mathrm{N}(y) = x \cdot \overline{y} + y \cdot \overline{x} = \mathrm{Tr}(x \cdot \overline{y})$.

Although the multiplication law of $\mathbb{O}_{\mathbb{R}}$ is not associative, it is still trace-associative in the sense that

$$\mathrm{Tr}((x \cdot y) \cdot z) = \mathrm{Tr}(x \cdot (y \cdot z)) \text{ for all } x, y, z \in \mathbb{O}_{\mathbb{R}},$$

and we can define $\mathrm{Tr}(xyz) := \mathrm{Tr}((x \cdot y) \cdot z) = \mathrm{Tr}(x \cdot (y \cdot z))$.

For our construction, we still have to recall the exceptional Jordan algebra, following [Con12, §6]:

Definition 2.1.3. The *(positive definite) real exceptional Jordan algebra*, denoted by $\mathrm{J}_{\mathbb{R}}$, is the 27-dimensional $\mathbb{R}$-vector space consisting of "Hermitian" matrices in $\mathrm{M}_3(\mathbb{O}_{\mathbb{R}})$, i.e. matrices of the form

$$\begin{pmatrix} a & z & \overline{y} \\ \overline{z} & b & x \\ y & \overline{x} & c \end{pmatrix}, \ a, b, c \in \mathbb{R}, \ x, y, z \in \mathbb{O}_{\mathbb{R}},$$

equipped with the $\mathbb{R}$-bilinear multiplication law

$$\mathrm{J}_{\mathbb{R}} \times \mathrm{J}_{\mathbb{R}} \to \mathrm{J}_{\mathbb{R}}, A \circ B := \frac{1}{2}(AB + BA),$$

where AB and BA denote the usual product of octonionic matrices, and with 2-sided identity element I given by the standard matrix identity element $\mathrm{diag}(1, 1, 1)$.

As an $\mathbb{R}$-algebra, $\mathrm{J}_{\mathbb{R}}$ is commutative but not associative.

Notation 2.1.4. To compress the space, when we do not need to emphasize the matrix structure of elements in $\mathrm{J}_{\mathbb{R}}$, we denote the element

$$\begin{pmatrix} a & z & \overline{y} \\ \overline{z} & b & x \\ y & \overline{x} & c \end{pmatrix}, \ a, b, c \in \mathbb{R}, \ x, y, z \in \mathbb{O}_{\mathbb{R}}$$

by $[a, b, c\,; x, y, z]$ for short.

The *trace* of $A = [a, b, c\,; x, y, z] \in \mathrm{J}_{\mathbb{R}}$ is defined as $\mathrm{Tr}(A) := a + b + c$. The underlying vector space of $\mathrm{J}_{\mathbb{R}}$ is equipped with the non-degenerate positive definite quadratic form:

$$\mathrm{Q}(A) := \mathrm{Tr}(A \circ A)/2 = \frac{1}{2}(a^2 + b^2 + c^2) + \mathrm{N}(x) + \mathrm{N}(y) + \mathrm{N}(z). \tag{2.2}$$

Its associated bilinear form is $\mathrm{B_Q}(A,B) := \mathrm{Q}(A+B) - \mathrm{Q}(A) - \mathrm{Q}(B) = \mathrm{Tr}(A \circ B)$. The *determinant* of the matrix A is defined by

$$\det(A) := abc + \mathrm{Tr}(xyz) - a\mathrm{N}(x) - b\mathrm{N}(y) - c\mathrm{N}(z). \tag{2.3}$$

It defines a cubic form on $\mathrm{J}_{\mathbb{R}}$.

We denote by F_4 the subgroup $\mathrm{Aut}(\mathrm{J}_{\mathbb{R}}, \circ)$ of $\mathrm{GL}(\mathrm{J}_{\mathbb{R}})$ consisting of elements $g \in \mathrm{GL}(\mathrm{J}_{\mathbb{R}})$ such that for all $A, B \in \mathrm{J}_{\mathbb{R}}$, $g(A \circ B) = g(A) \circ g(B)$. It is a compact Lie group of type F_4 [Ada96, Theorem 16.7].

In this paper, we deal with automorphic forms so we want a reductive group over $\mathbb{Q}$ whose real points is isomorphic to F_4. For this purpose, we first define the following $\mathbb{Q}$-algebras:

Definition 2.1.5. *Cayley's definite octonion algebra* $\mathbb{O}_{\mathbb{Q}}$ is the sub-$\mathbb{Q}$-algebra of $\mathbb{O}_{\mathbb{R}}$ generated by $\{\mathrm{e}_1, \ldots, \mathrm{e}_7\}$. The *(positive definite) rational exceptional Jordan algebra* $\mathrm{J}_{\mathbb{Q}}$ is the sub-$\mathbb{Q}$-space of $\mathrm{J}_{\mathbb{R}}$ consisting of $[a,b,c\,;x,y,z], a,b,c \in \mathbb{Q}, x,y,z \in \mathbb{O}_{\mathbb{Q}}$ equipped with the multiplication $\circ$.

The main object considered in this paper is the following algebraic group:

Definition 2.1.6. We define $\mathbf{F}_4$ to be the closed subgroup of the algebraic $\mathbb{Q}$-group $\mathrm{GL}_{\mathrm{J}_{\mathbb{Q}}}$, which as a functor sends a commutative unital $\mathbb{Q}$-algebra R to the group

$$\mathbf{F}_4(R) := \mathrm{Aut}(\mathrm{J}_{\mathbb{Q}} \otimes_{\mathbb{Q}} R, \circ) = \{g \in \mathrm{GL}(\mathrm{J}_{\mathbb{Q}} \otimes_{\mathbb{Q}} R) \,|\, g(A \circ B) = g(A) \circ g(B), \forall A, B \in \mathrm{J}_{\mathbb{Q}} \otimes_{\mathbb{Q}} R\}.$$

From the definition we have $\mathbf{F}_4(\mathbb{R}) = \mathrm{F}_4$. By [Spr00, Theorem 7.2.1], $\mathbf{F}_4$ is a semisimple and simply-connected group over $\mathbb{Q}$.

Remark 2.1.7. We have an alternative description of $\mathbf{F}_4$ that we will use later: the closed subgroup $\mathrm{Aut}_{(\mathrm{J}_{\mathbb{Q}}, \det, \mathrm{I})/\mathbb{Q}}$ of $\mathrm{GL}_{\mathrm{J}_{\mathbb{Q}}}$ consisting of linear automorphisms that preserve both the cubic form det and the identity element I. The closed subgroups $\mathbf{F}_4 = \mathrm{Aut}_{(\mathrm{J}_{\mathbb{Q}}, \circ)/\mathbb{Q}}$ and $\mathrm{Aut}_{(\mathrm{J}_{\mathbb{Q}}, \det, \mathrm{I})/\mathbb{Q}}$ inside $\mathrm{GL}_{\mathrm{J}_{\mathbb{Q}}}$ are both smooth and they have the same geometric points according to [Spr00, Proposition 5.9.4], so they coincide.

2.2 Reductive $\mathbb{Z}$-models of reductive $\mathbb{Q}$-groups

Now we recall some results in [Gro96; Gro99b]. In this subsection, let G be a connected reductive algebraic group over $\mathbb{Q}$. Denote the product $\prod_p \mathbb{Z}_p$ by $\widehat{\mathbb{Z}}$ and let $\mathbb{A}_f = \widehat{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Q}$ be the ring of finite adèles, and $\mathbb{A} = \mathbb{R} \times \mathbb{A}_f$.

Definition 2.2.1. A *reductive $\mathbb{Z}$-model* of G is a pair $(\mathscr{G}, \iota)$ consisting of:

- an affine smooth group scheme $\mathscr{G}$ of finite type over $\mathbb{Z}$ such that $\mathscr{G} \otimes_{\mathbb{Z}} \mathbb{Z}/p\mathbb{Z}$ is reductive over $\mathbb{Z}/p\mathbb{Z}$ for each prime number p,
- an isomorphism $\iota : \mathscr{G} \otimes_{\mathbb{Z}} \mathbb{Q} \simeq G$ of algebraic groups over $\mathbb{Q}$.

Two reductive $\mathbb{Z}$-models $(\mathscr{G}_1, \iota_1)$ and $(\mathscr{G}_2, \iota_2)$ are said to be isomorphic if there exists an isomorphism $f : \mathscr{G}_1 \to \mathscr{G}_2$ over $\mathbb{Z}$ such that the following diagram commutes:

$$\begin{tikzcd} \mathscr{G}_1 \otimes_{\mathbb{Z}} \mathbb{Q} \arrow[rr, "f_{\mathbb{Q}}"] \arrow[dr, "\iota_1"'] & & \mathscr{G}_2 \otimes_{\mathbb{Z}} \mathbb{Q} \arrow[dl, "\iota_2"] \\ & G & \end{tikzcd}$$

Remark 2.2.2. When there is no confusion about ι, we simply say that $\mathscr{G}$ is a reductive $\mathbb{Z}$-model of G.

From the theory of *Chevalley groups* in [SGA3, §XXV], every group G split over $\mathbb{Q}$ admits a reductive $\mathbb{Z}$-model. Indeed, we can take the Chevalley group with the same root datum of G to be its reductive $\mathbb{Z}$-model.

When G is not split, in general the existence of reductive $\mathbb{Z}$-models of G is no longer ensured. Now we consider the case when G is *anisotropic*, i.e. G does not contain any non-trivial split $\mathbb{Q}$-torus. When G has a reductive $\mathbb{Z}$-model, being anisotropic is equivalent to that $G(\mathbb{R})$ is compact, which is due to [PR94, Theorem 5.5(1)] and [Gro96, Proposition 2.1]. In [Gro96, §1], Gross proves the following result:

Theorem 2.2.3. *Let G be an anisotropic semisimple simply-connected $\mathbb{Q}$-group such that the root system of $G_{\mathbb{C}}$ is irreducible, then G admits a reductive $\mathbb{Z}$-model if and only if the Lie type of G is among:*

$$\mathrm{B}_{(d-1)/2}\,(d \equiv \pm 0 \bmod 8), \mathrm{D}_{d/2}\,(d \equiv 1 \bmod 8), \mathrm{G}_2, \mathrm{F}_4, \mathrm{E}_8.$$

The next question is to classify reductive $\mathbb{Z}$-models of a given anisotropic group G up to some equivalence relation.

Definition 2.2.4. Let $(\mathscr{G}, \mathrm{id})$ be a reductive $\mathbb{Z}$-model of its generic fiber $G := \mathscr{G} \otimes_{\mathbb{Z}} \mathbb{Q}$. A reductive $\mathbb{Z}$-model $(\mathscr{G}', \iota')$ of G is said to be in the same *genus* as $\mathscr{G}$, if $\iota'(\mathscr{G}'(\widehat{\mathbb{Z}}))$ and $\mathscr{G}(\widehat{\mathbb{Z}})$ are conjugate in $G(\mathbb{A}_f)$.

Remark 2.2.5. This condition is equivalent to that for each prime p, $\iota'(\mathscr{G}'(\mathbb{Z}_p))$ is conjugate to $\mathscr{G}(\mathbb{Z}_p)$ in $G(\mathbb{Q}_p)$, and $\iota'(\mathscr{G}'(\mathbb{Z}_p)) = \mathscr{G}(\mathbb{Z}_p)$ for almost all p.

By [Gro99b, Proposition 1.4], the equivalence classes of reductive $\mathbb{Z}$-models in the genus of $\mathscr{G}$ can be identified with the coset space $G(\mathbb{A}_f)/\mathscr{G}(\widehat{\mathbb{Z}})$.

The group $G(\mathbb{Q})$ acts on reductive $\mathbb{Z}$-models in the genus of $\mathscr{G}$ by the formula:

$$g(\mathscr{G}', \iota') = (\mathscr{G}', \mathrm{ad}(g) \circ \iota'),$$

where $\mathrm{ad}(g)$ is the conjugation by g. This induces an action of $G(\mathbb{Q})$ on the equivalence classes of reductive $\mathbb{Z}$-models in the genus of $\mathscr{G}$. We say two reductive $\mathbb{Z}$-models in the genus of $\mathscr{G}$ are $G(\mathbb{Q})$-*conjugate* if their equivalence classes are in the same $G(\mathbb{Q})$-orbit.

Now the set of $G(\mathbb{Q})$-orbits on the equivalence classes of reductive $\mathbb{Z}$-models in the genus of $\mathscr{G}$ can be identified with the double coset space $G(\mathbb{Q})\backslash G(\mathbb{A}_f)/\mathscr{G}(\widehat{\mathbb{Z}})$, which is finite by Borel's famous result [Bor63].

2.3 Reductive $\mathbb{Z}$-models of F_4

For our $\mathbb{Q}$-group $\mathbf{F}_4$, the $\mathbf{F}_4(\mathbb{Q})$-orbits of equivalence classes of reductive $\mathbb{Z}$-models of $\mathbf{F}_4$ in some genus is determined by Gross in [Gro96, Proposition 5.3], using the mass formula [Gro96, Proposition 2.2]. In this subsection we provide an alternative proof for his result, which will be helpful for our computations in §3.

2.3.1 Integral structures of $\mathbb{O}_\mathbb{Q}$ and $\mathrm{J}_\mathbb{Q}$

Parallel to the construction of $\mathbf{F}_4$ in §2.1, we want to define integral structures of $\mathbb{O}_\mathbb{Q}$ and $\mathrm{J}_\mathbb{Q}$ and then use them to construct reductive $\mathbb{Z}$-models of $\mathbf{F}_4$.

Definition 2.3.1. *Coxeter's integral order* $\mathbb{O}_\mathbb{Z}$ is the $\mathbb{Z}$-lattice of rank 8 inside $\mathbb{O}_\mathbb{Q}$ spanned by the lattice $\mathbb{Z} \oplus \mathbb{Z}\mathrm{e}_1 \oplus \cdots \oplus \mathbb{Z}\mathrm{e}_7$ and the four elements

$$\mathrm{h}_1 = (1 + \mathrm{e}_1 + \mathrm{e}_2 + \mathrm{e}_4)/2, \mathrm{h}_2 = (1 + \mathrm{e}_1 + \mathrm{e}_3 + \mathrm{e}_7)/2,$$
$$\mathrm{h}_3 = (1 + \mathrm{e}_1 + \mathrm{e}_5 + \mathrm{e}_6)/2, \mathrm{h}_4 = (\mathrm{e}_1 + \mathrm{e}_2 + \mathrm{e}_3 + \mathrm{e}_5)/2,$$

equipped with the multiplication of $\mathbb{O}_\mathbb{Q}$. This lattice contains the identity element of $\mathbb{O}_\mathbb{Q}$ and is stable under the multiplication, i.e. is an *order* in $\mathbb{O}_\mathbb{Q}$.

Remark 2.3.2. The underlying lattice of $\mathbb{O}_\mathbb{Z}$ equipped with the quadratic form $\mathrm{N}|_{\mathbb{O}_\mathbb{Z}}$ is isometric to the even unimodular lattice

$$\mathrm{E}_8 = \left\{ (x_i) \in \mathbb{Z}^8 \cup (\mathbb{Z} + \frac{1}{2})^8 \,\middle|\, \sum_i x_i \equiv 0 \bmod 2 \right\}.$$

Let $\mathrm{J}_\mathbb{Z}$ be the lattice

$$\{[a, b, c\,; x, y, z] \in \mathrm{J}_\mathbb{Q} \,|\, a, b, c \in \mathbb{Z}, x, y, z \in \mathbb{O}_\mathbb{Z}\}$$

of rank 27 inside the $\mathbb{Q}$-vector space $\mathrm{J}_\mathbb{Q}$. This lattice is stable under the Jordan multiplication $\circ$ on $\mathrm{J}_\mathbb{Q}$, thus $\mathrm{J}_\mathbb{Z}$ is an order in $\mathrm{J}_\mathbb{Q}$.

As in Remark 2.1.7, the $\mathbb{Q}$-group $\mathbf{F}_4$ coincides with the group $\mathrm{Aut}_{(\mathrm{J}_\mathbb{Q},\det,\mathrm{I})/\mathbb{Q}}$. The triple $(\mathrm{J}_\mathbb{Q}, \det, \mathrm{I})$ has a natural integral structure $(\mathrm{J}_\mathbb{Z}, \det, \mathrm{I})$. The $\mathbb{Z}$-group scheme $\mathrm{Aut}_{(\mathrm{J}_\mathbb{Z},\det,\mathrm{I})/\mathbb{Z}}$, sending any commutative $\mathbb{Z}$-algebra R to the subgroup of $\mathrm{GL}(\mathrm{J}_\mathbb{Z} \otimes_\mathbb{Z} R)$ consisting of elements preserve the cubic form det and the identity element I, is expected to be a reductive $\mathbb{Z}$-model of $\mathbf{F}_4$, but we are going to consider the $\mathbb{Z}$-group scheme $\mathrm{Aut}_{(\mathrm{J}_\mathbb{Z},\det,e)/\mathbb{Z}}$ for any $e \in \mathrm{J}_\mathbb{Z}$ satisfying certain conditions, in order to produce several reductive $\mathbb{Z}$-models of $\mathbf{F}_4$ uniformly.

Definition 2.3.3. An element

$$A = \begin{pmatrix} a & z & \overline{y} \\ \overline{z} & b & x \\ y & \overline{x} & c \end{pmatrix} \in \mathrm{J}_\mathbb{R}$$

is said to be *positive definite* if its seven "minor determinants"

$$a, b, c, ab - \mathrm{N}(z), bc - \mathrm{N}(x), ca - \mathrm{N}(y), \det(A) \in \mathbb{R}$$

are all positive. A positive definite element e in $\mathrm{J}_\mathbb{R}$ with $\det e = 1$ is called a *polarization*.

Given a polarization e contained in the lattice $\mathrm{J}_\mathbb{Z}$, one constructs a $\mathbb{Z}$-group scheme $\mathcal{F}_{4,e} := \mathrm{Aut}_{(\mathrm{J}_\mathbb{Z},\det,e)/\mathbb{Z}}$ in the same way as $\mathrm{Aut}_{(\mathrm{J}_\mathbb{Z},\det,\mathrm{I})/\mathbb{Z}}$. The following result shows that this group scheme is a reductive $\mathbb{Z}$-model of $\mathbf{F}_4$.

Proposition 2.3.4. *[Con12, Proposition 6.6, Example 6.7] For any choice of polarization $e \in \mathrm{J}_{\mathbb{Z}}$, the fiber $\mathcal{F}_{4,e} \otimes_{\mathbb{Z}} \mathbb{Z}/p\mathbb{Z}$ is semisimple for every prime number p, and $\mathcal{F}_{4,e}(\mathbb{R})$ is a compact Lie group of type* F_4.

Taking e to be the identity element I, the generic fiber of $\mathcal{F}_{4,\mathrm{I}}$ is $\mathrm{Aut}_{(\mathrm{J}_{\mathbb{Q}},\det,\mathrm{I})/\mathbb{Q}} = \mathbf{F}_4$, thus $\mathcal{F}_{4,\mathrm{I}}$ is a reductive $\mathbb{Z}$-model of $\mathbf{F}_4$.

If we take e to be

$$\mathrm{E} := [2,2,2\,;\beta,\beta,\beta],\beta = \frac{1}{2}(-1+\mathrm{e}_1+\mathrm{e}_2+\cdots+\mathrm{e}_7) \in \mathrm{J}_{\mathbb{Z}},$$

as in [EG96, (5.4)], by [Con12, Example 6.7] the generic fiber of $\mathcal{F}_{4,\mathrm{E}}$ is isomorphic to $\mathbf{F}_4$. We denote the natural isomorphism $\mathcal{F}_{4,\mathrm{E}} \otimes_{\mathbb{Z}} \mathbb{Q} \to \mathbf{F}_4$ by ι. Actually ι can be given as the conjugation by an element in the $\mathbb{Q}$-points of the $\mathbb{Q}$-group $\mathrm{Aut}_{(\mathrm{J}_{\mathbb{Q}},\det)/\mathbb{Q}}$ which sends E to I.

In [Gro96, Proposition 5.3], Gross proves the following result:

Proposition 2.3.5. *There are two $\mathbf{F}_4(\mathbb{Q})$-orbits on the equivalence classes of reductive $\mathbb{Z}$-models of $\mathbf{F}_4$ in the genus of $\mathcal{F}_{4,\mathrm{I}}$, whose representatives are given by $(\mathcal{F}_{4,\mathrm{I}},\mathrm{id})$ and $(\mathcal{F}_{4,\mathrm{E}},\iota)$ respectively.*

Applying the mass formula [Gro96, Proposition 2.2] to $\mathbf{F}_4$, we have

$$\sum_{(\mathscr{G},\iota)} \frac{1}{|\mathscr{G}(\mathbb{Z})|} = \frac{1}{2^4}\zeta(-1)\zeta(-5)\zeta(-7)\zeta(-11) = \frac{691}{2^{15}\cdot 3^6\cdot 5^2\cdot 7^2\cdot 13}, \tag{2.4}$$

where $(\mathscr{G},\iota)$ varies over the $\mathbf{F}_4(\mathbb{Q})$-conjugacy classes of reductive $\mathbb{Z}$-models of $\mathbf{F}_4$ in the genus of $\mathcal{F}_{4,\mathrm{I}}$. As

$$\frac{691}{2^{15}\cdot 3^6\cdot 5^2\cdot 7^2\cdot 13} = \frac{1}{2^{15}\cdot 3^6\cdot 5^2\cdot 7} + \frac{1}{2^{12}\cdot 3^5\cdot 7^2\cdot 13}, \tag{2.5}$$

in order to prove Proposition 2.3.5 it suffices to prove the following two things:

- $\mathcal{F}_{4,\mathrm{I}}$ and $\mathcal{F}_{4,\mathrm{E}}$ are not $\mathbf{F}_4(\mathbb{Q})$-conjugate.
- $|\mathcal{F}_{4,\mathrm{I}}(\mathbb{Z})| \leq 2^{15}\cdot 3^6\cdot 5^2\cdot 7$ and $|\mathcal{F}_{4,\mathrm{E}}(\mathbb{Z})| \leq 2^{12}\cdot 3^5\cdot 7^2\cdot 13$.

In his proof, Gross cites some results from [ATLAS], We are going to give another proof of Proposition 2.3.5, which avoids using results in [ATLAS].

2.3.2 $\mathcal{F}_{4,\mathrm{E}}(\mathbb{Z})$

Now we deal with the finite group $\mathcal{F}_{4,\mathrm{E}}(\mathbb{Z})$. Our goal is to prove:

Proposition 2.3.6. $|\mathcal{F}_{4,\mathrm{E}}(\mathbb{Z})| \leq 2^{12}\cdot 3^5\cdot 7^2\cdot 13$.

With the choice of polarization E, we can define a new bilinear form on $\mathrm{J}_{\mathbb{Q}}$:

$$\langle A,B\rangle_{\mathrm{E}} = (A,\mathrm{E},\mathrm{E})(B,\mathrm{E},\mathrm{E}) - 2(A,B,\mathrm{E}),$$

where the trilinear form $(\ ,\ ,\ ) : \mathrm{J}_{\mathbb{Q}}^3 \to \mathbb{Q}$ is defined by

$$\begin{aligned}(A,B,C) =& \frac{1}{2}[\det(A+B+C) - \det(A+B) - \det(B+C) - \det(C+A)\\ &+ \det(A) + \det(B) + \det(C)].\end{aligned}$$

This bilinear form is positive definite and integral on $\mathrm{J}_{\mathbb{Z}}$ by [EG96, Proposition 7.2].

Notation 2.3.7. Here we give some notations for elements in $\mathrm{J}_{\mathbb{R}}$: we write

$$\mathrm{E}_1 := [1,0,0\,;0,0,0], \mathrm{E}_2 := [0,1,0\,;0,0,0], \mathrm{E}_3 := [0,0,1\,;0,0,0]$$

and for any $x \in \mathbb{O}_{\mathbb{R}}$,

$$\mathrm{F}_1(x) := [0,0,0\,;x,0,0], \mathrm{F}_2(x) := [0,0,0\,;0,x,0], \mathrm{F}_3(x) := [0,0,0\,;0,0,x].$$

Note that $1, \mathrm{e}_1, \mathrm{e}_2, \mathrm{e}_3, \mathrm{h}_1, \mathrm{h}_2, \mathrm{h}_3, \mathrm{h}_4$ is a basis of the lattice $\mathbb{O}_{\mathbb{Z}}$, thus we have the following basis of $\mathrm{J}_{\mathbb{Z}}$:

$$\mathcal{B} := \left(\begin{array}{c} \mathrm{E}_1, \mathrm{E}_2, \mathrm{E}_3, \mathrm{F}_1(1), \mathrm{F}_1(\mathrm{e}_1), \mathrm{F}_1(\mathrm{e}_2), \mathrm{F}_1(\mathrm{e}_3), \mathrm{F}_1(\mathrm{h}_1), \mathrm{F}_1(\mathrm{h}_2), \mathrm{F}_1(\mathrm{h}_3), \mathrm{F}_1(\mathrm{h}_4), \mathrm{F}_2(1), \mathrm{F}_2(\mathrm{e}_1), \mathrm{F}_2(\mathrm{e}_2), \\ \mathrm{F}_2(\mathrm{e}_3), \mathrm{F}_2(\mathrm{h}_1), \mathrm{F}_2(\mathrm{h}_2), \mathrm{F}_2(\mathrm{h}_3), \mathrm{F}_2(\mathrm{h}_4), \mathrm{F}_3(1), \mathrm{F}_3(\mathrm{e}_1), \mathrm{F}_3(\mathrm{e}_2), \mathrm{F}_3(\mathrm{e}_3), \mathrm{F}_3(\mathrm{h}_1), \mathrm{F}_3(\mathrm{h}_2), \mathrm{F}_3(\mathrm{h}_3), \mathrm{F}_3(\mathrm{h}_4) \end{array} \right). \tag{2.6}$$

In the basis $\mathcal{B}$, we give the Gram matrix of the quadratic lattice $(\mathrm{J}_{\mathbb{Z}}, \langle\ ,\ \rangle_{\mathrm{E}})$ in Figure 1, Appendix A.

Proof of Proposition 2.3.6. Each element in $\mathcal{F}_{4,\mathrm{E}}(\mathbb{Z}) = \mathrm{Aut}(\mathrm{J}_{\mathbb{Z}}, \det, \mathrm{E})$ preserves the bilinear form $\langle\ ,\ \rangle_{\mathrm{E}}$ by the definition, thus this finite group is a subgroup of the isometry group $\mathrm{O}(\mathrm{J}_{\mathbb{Z}}, \langle\ ,\ \rangle_{\mathrm{E}})$ of the quadratic lattice $(\mathrm{J}_{\mathbb{Z}}, \langle\ ,\ \rangle_{\mathrm{E}})$.

The order of $\mathrm{O}(\mathrm{J}_{\mathbb{Z}}, \langle\ ,\ \rangle_{\mathrm{E}})$ can be determined with the help of the Plesken-Souvignier algorithm. Concretely, we can apply the `qfauto` function in [PARI/GP] to the Gram matrix Figure 1 of $(\mathrm{J}_{\mathbb{Z}}, \langle\ ,\ \rangle_{\mathrm{E}})$, and we find

$$|\mathrm{O}(\mathrm{J}_{\mathbb{Z}}, \langle\ ,\ \rangle_{\mathrm{E}})| = 2^{13} \cdot 3^5 \cdot 7^2 \cdot 13.$$

Notice that the isometry group contains an involution $-\mathrm{id}$, which does not fix E, thus we have

$$|\mathcal{F}_{4,\mathrm{E}}(\mathbb{Z})| \leq \frac{1}{2}\, |\mathrm{O}(\mathrm{J}_{\mathbb{Z}}, \langle\ ,\ \rangle_{\mathrm{E}})| = 2^{12} \cdot 3^5 \cdot 7^2 \cdot 13.$$

□

Remark 2.3.8. The orthogonal complement of E in $(\mathrm{J}_{\mathbb{Z}}, \langle\ ,\ \rangle_{\mathrm{E}})$ is a 26-dimensional even lattice of determinant 3 and with no roots [EG96, Proposition 7.2]. In Borcherds' thesis [Bor99, §5.7], he proves that a lattice satisfying these conditions is unique up to isomorphism and calculates the order of its isometry group, giving another proof of Proposition 2.3.6.

Furthermore, the `qfauto` function also give us a set of generators $\{-\mathrm{id}, -\sigma_1, \sigma_2\}$ of $\mathrm{O}(\mathrm{J}_{\mathbb{Z}}, \langle\ ,\ \rangle_{\mathrm{E}})$, where the matrices of σ_1, σ_2 in the basis $\mathcal{B}$ (2.6) are given in Figure 2, Appendix A. Here we write $-\sigma_1$ instead of σ_1 because the second element in the result given by [PARI/GP] sends E to $-\mathrm{E}$. The isometry group $\mathrm{O}(\mathrm{J}_{\mathbb{Z}}, \langle\ ,\ \rangle_{\mathrm{E}})$ is the direct product of the subgroup generated by σ_1, σ_2 and the order 2 central subgroup $\pm\mathrm{id}$. In the proof of Proposition 2.3.6, we find that $\mathcal{F}_{4,\mathrm{E}}(\mathbb{Z})$ is a subgroup of the group $\langle \sigma_1, \sigma_2 \rangle$.

In the basis $\mathcal{B}$, the cubic form det on $\mathrm{J}_{\mathbb{R}}$ can be written down as a 27-variable polynomial of degree 3, and we give this polynomial function as `MatDet` in our [PARI/GP] program [Sha]. Using [PARI/GP], we verify that σ_1 and σ_2 both preserve the cubic form det and the element E, thus $\mathcal{F}_{4,\mathrm{E}}(\mathbb{Z})$ and the group $\langle \sigma_1, \sigma_2 \rangle$ coincide and $|\mathcal{F}_{4,\mathrm{E}}(\mathbb{Z})| = 2^{12} \cdot 3^5 \cdot 7^2 \cdot 13$.

2.3.3 $\mathcal{F}_{4,\mathrm{I}}(\mathbb{Z})$

Now we look at the finite group $\mathcal{F}_{4,\mathrm{I}}(\mathbb{Z}) = \mathrm{Aut}(\mathrm{J}_\mathbb{Z}, \det, \mathrm{I}) = \mathrm{Aut}(\mathrm{J}_\mathbb{Z}, \circ)$, and we want to prove the following proposition:

Proposition 2.3.9. *The reductive $\mathbb{Z}$-model $\mathcal{F}_{4,\mathrm{I}}$ of $\mathbf{F}_4$ is not $\mathbf{F}_4(\mathbb{Q})$-conjugate to $\mathcal{F}_{4,\mathrm{E}}$, and* $|\mathcal{F}_{4,\mathrm{I}}(\mathbb{Z})| \leq 2^{15} \cdot 3^6 \cdot 5^2 \cdot 7$.

Denote the subset of $\mathrm{J}_\mathbb{Z}$ consisting of diagonal matrices by D, and the subset of elements whose diagonal entries are zero by D_0. The formula (2.2) for the quadratic form Q on $\mathrm{J}_\mathbb{Z}$ shows that equipped with Q we have $\mathrm{J}_\mathbb{Z} = \mathrm{D}_0 \oplus \mathrm{D}$ as quadratic lattices. By Remark 2.3.2, the quadratic lattice $(\mathbb{O}_\mathbb{Z}, \mathrm{N})$ is isometric to E_8, thus D_0 is isometric to $\mathrm{E}_8 \oplus \mathrm{E}_8 \oplus \mathrm{E}_8$. On the other hand, the lattice D is isometric to

$$\mathrm{I}_3 = \mathbb{Z}^3, \mathrm{q} : (x_1, x_2, x_3) \mapsto \frac{1}{2}\left(x_1^2 + x_2^2 + x_3^2\right).$$

Any element of $\mathcal{F}_{4,\mathrm{I}}(\mathbb{Z})$ preserves the quadratic form Q on $\mathrm{J}_\mathbb{Z}$, so $\mathcal{F}_{4,\mathrm{I}}(\mathbb{Z})$ is a subgroup of the isometry group $\mathrm{O}(\mathrm{J}_\mathbb{Z})$ of the quadratic lattice $\mathrm{J}_\mathbb{Z}$. By the theory of root lattices, we have

$$\mathrm{O}(\mathrm{J}_\mathbb{Z}) \simeq \mathrm{O}(\mathrm{I}_3) \times \left(\mathrm{O}(\mathbb{O}_\mathbb{Z}) \wr \mathrm{S}_3\right),$$

where S_3 is the permutation group of three elements and $\wr$ stands for the wreath product. Let p be the restriction map $\mathcal{F}_{4,\mathrm{I}}(\mathbb{Z}) \hookrightarrow \mathrm{O}(\mathrm{J}_\mathbb{Z}) \twoheadrightarrow \mathrm{O}(\mathrm{D}), g \mapsto g|_\mathrm{D}$, where $\mathrm{O}(\mathrm{D}) \simeq \mathrm{O}(\mathrm{I}_3)$ is isomorphic to $\{\pm 1\}^3 \rtimes \mathrm{S}_3$.

Let $\mathrm{O}(\mathrm{D}\,;\mathrm{I})$ be the group $\{\sigma \in \mathrm{O}(\mathrm{D}) \,|\, \sigma(\mathrm{I}) = \mathrm{I}\}$, which is isomorphic to the permutation group S_3. Since elements in $\mathcal{F}_{4,\mathrm{I}}(\mathbb{Z})$ fix I, the image of p is contained in $\mathrm{O}(\mathrm{D}\,;\mathrm{I})$.

Lemma 2.3.10. *The image of p is* $\mathrm{O}(\mathrm{D}\,;\mathrm{I}) \simeq \mathrm{S}_3$.

Proof. For an element $\sigma \in \mathrm{S}_3$, we denote by g_σ the element

$$[a_1, a_2, a_3\,; x_1, x_2, x_3] \mapsto [a_{\sigma^{-1}(1)}, a_{\sigma^{-1}(2)}, a_{\sigma^{-1}(3)}\,; \epsilon(\sigma)(x_{\sigma^{-1}(1)}), \epsilon(\sigma)(x_{\sigma^{-1}(2)}), \epsilon(\sigma)(x_{\sigma^{-1}(3)})] \tag{2.7}$$

in $\mathrm{GL}(\mathrm{J}_\mathbb{Z})$, where the map $\epsilon(\sigma) : \mathbb{O}_\mathbb{Z} \to \mathbb{O}_\mathbb{Z}$ is defined as identity when σ is even, and as the conjugation when σ is odd. In this proof, we write $x^* := \epsilon(\sigma)(x)$ for short.

For any $A = [a_1, a_2, a_3\,; x_1, x_2, x_3] \in \mathrm{J}_\mathbb{Z}$, by the formula (2.3) for the cubic form det, we have

$$\begin{aligned}\det\left(g_\sigma(A)\right) =& \prod_{i=1}^3 a_{\sigma^{-1}(i)} + \mathrm{Tr}(x^*_{\sigma^{-1}(1)} x^*_{\sigma^{-1}(2)} x^*_{\sigma^{-1}(3)}) - \sum_{i=1}^3 a_{\sigma^{-1}(i)} \mathrm{N}(x^*_{\sigma^{-1}(i)}) \\ =& a_1 a_2 a_3 + \mathrm{Tr}(x^*_{\sigma^{-1}(1)} x^*_{\sigma^{-1}(2)} x^*_{\sigma^{-1}(3)}) - \sum_{i=1}^3 a_i \mathrm{N}(x_i).\end{aligned}$$

The property (2.1) of Tr implies that for any $x, y, z \in \mathbb{O}_\mathbb{Z}$,

$$\mathrm{Tr}(xyz) = \mathrm{Tr}(yzx) = \mathrm{Tr}(zxy) = \mathrm{Tr}(\overline{x} \cdot \overline{z} \cdot \overline{y}) = \mathrm{Tr}(\overline{z} \cdot \overline{y} \cdot \overline{x}) = \mathrm{Tr}(\overline{y} \cdot \overline{x} \cdot \overline{z}),$$

which can also be stated as $\mathrm{Tr}(x^*_{\sigma^{-1}(1)} x^*_{\sigma^{-1}(2)} x^*_{\sigma^{-1}(3)}) = \mathrm{Tr}(x_1 x_2 x_3)$ for any $\sigma \in \mathrm{S}_3$. Hence $\det(g_\sigma(A)) = \det(A)$. Since g_σ also fixes I, it is an element in $\mathcal{F}_{4,\mathrm{I}}(\mathbb{Z})$ and its restriction $p(g_\sigma) \in \mathrm{O}(\mathrm{D}\,;\mathrm{I}) \simeq \mathrm{S}_3$ is σ, thus $\mathrm{Im}(p) = \mathrm{O}(\mathrm{D}\,;\mathrm{I})$. $\square$

Let $\mathscr{D}$ be the kernel of p, then we have a short exact sequence of finite groups:

$$1 \to \mathscr{D} \to \mathcal{F}_{4,\mathrm{I}}(\mathbb{Z}) \to \mathrm{O}(\mathrm{D}\,;\mathrm{I}) \simeq \mathrm{S}_3 \to 1. \tag{2.8}$$

Lemma 2.3.11. *The map $\kappa : \mathrm{S}_3 \to \mathcal{F}_{4,\mathrm{I}}(\mathbb{Z}), \sigma \mapsto g_\sigma$ defined in (2.7) gives a splitting of the short exact sequence (2.8).*

Proof. It suffices to show that $\sigma \mapsto g_\sigma$ is a group homomorphism. For $\sigma, \tau \in \mathrm{S}_3$, we have

$$\begin{aligned}
&g_\tau \circ g_\sigma\left([a_1, a_2, a_3\,; x_1, x_2, x_3]\right) \\
=&g_\tau\left([a_{\sigma^{-1}(1)}, a_{\sigma^{-1}(2)}, a_{\sigma^{-1}(3)}\,; \epsilon(\sigma)(x_{\sigma^{-1}(1)}), \epsilon(\sigma)(x_{\sigma^{-1}(2)}), \epsilon(\sigma)(x_{\sigma^{-1}(3)})]\right) \\
=&\left[\begin{array}{c} a_{(\tau\sigma)^{-1}(1)}, a_{(\tau\sigma)^{-1}(2)}, a_{(\tau\sigma)^{-1}(3)}\,; \\ \epsilon(\tau)\circ\epsilon(\sigma)(x_{(\tau\sigma)^{-1}(1)}), \epsilon(\tau)\circ\epsilon(\sigma)(x_{(\tau\sigma)^{-1}(2)}), \epsilon(\tau)\circ\epsilon(\sigma)(x_{(\tau\sigma)^{-1}(3)}) \end{array}\right].
\end{aligned}$$

It can be easily seen that the map $\epsilon : \mathrm{S}_3 \to \mathrm{GL}(\mathbb{O}_\mathbb{Z})$ is a group homomorphism, thus $g_\tau \circ g_\sigma = g_{\tau\sigma}$ and $\sigma \mapsto g_\sigma$ is also a group homomorphism. $\square$

This lemma tells us $\mathcal{F}_{4,\mathrm{I}}(\mathbb{Z}) = \mathscr{D} \rtimes \kappa(\mathrm{S}_3)$ and $|\mathcal{F}_{4,\mathrm{I}}(\mathbb{Z})| = 3! \cdot |\mathscr{D}|$. Now we study the structure of $\mathscr{D}$.

Lemma 2.3.12. *The group $\mathscr{D}$ is isomorphic to the group*

$$\widetilde{\mathrm{SO}(\mathbb{O}_\mathbb{Z})} := \left\{(\alpha, \beta, \gamma) \in \mathrm{SO}(\mathbb{O}_\mathbb{Z})^3 \,\middle|\, \overline{\alpha(x)\beta(y)} = \gamma(\overline{xy}), \forall x, y \in \mathbb{O}_\mathbb{Z}\right\}.$$

Proof. Fix $g \in \mathscr{D}$ and $x \in \mathbb{O}_\mathbb{Z}$, we define $y, z, w \in \mathbb{O}_\mathbb{Z}$ by the formula

$$g.\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & x \\ 0 & \overline{x} & 0 \end{pmatrix} = \begin{pmatrix} 0 & w & \overline{z} \\ \overline{w} & 0 & y \\ z & \overline{y} & 0 \end{pmatrix}.$$

Since g preserves the Jordan multiplication $\circ$, we have

$$\begin{aligned}
\mathrm{N}(x)\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} &= g.\left(\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & x \\ 0 & \overline{x} & 0 \end{pmatrix} \circ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & x \\ 0 & \overline{x} & 0 \end{pmatrix}\right) \\
&= \begin{pmatrix} 0 & w & \overline{z} \\ \overline{w} & 0 & y \\ z & \overline{y} & 0 \end{pmatrix} \circ \begin{pmatrix} 0 & w & \overline{z} \\ \overline{w} & 0 & y \\ z & \overline{y} & 0 \end{pmatrix} \\
&= \begin{pmatrix} \mathrm{N}(z)+\mathrm{N}(w) & \overline{yz} & wy \\ yz & \mathrm{N}(w)+\mathrm{N}(y) & \overline{zw} \\ \overline{wy} & zw & \mathrm{N}(y)+\mathrm{N}(z) \end{pmatrix},
\end{aligned}$$

which implies that $z = w = 0$ and y $= \mathrm{N}(x)$. This gives us a homomorphism $g \mapsto \alpha_g$ from $\mathscr{D}$ to $\mathrm{O}(\mathbb{O}_\mathbb{Z})$ such that $g[0,0,0\,;x,0,0] = [0,0,0\,;\alpha_g(x),0,0]$ for $\in \mathbb{O}_\mathbb{Z}$.

Symmetrically, we also get $\beta_g, \gamma_g \in \mathrm{O}(\mathbb{O}_\mathbb{Z})$ such that

$$g[0,0,0\,;x,y,z] = [0,0,0\,;\alpha_g(x),\beta_g(x),\gamma_g(x)] \text{ for all } x,y,z \in \mathbb{O}_\mathbb{Z}.$$

Taking determinants of both sides, we get

$$\mathrm{Tr}(xyz) = \mathrm{Tr}(\alpha_g(x)\beta_g(y)\gamma_g(z)) \text{ for all } x, y, z \in \mathbb{O}_{\mathbb{Z}}.$$

This is equivalent to $\langle \overline{\alpha_g(x)\beta_g(y)}, \gamma_g(z)\rangle = \langle \overline{xy}, z\rangle$. Since $\langle \overline{xy}, z\rangle = \langle \gamma_g(\overline{xy}), \gamma_g(z)\rangle$, we have

$$\langle \overline{\alpha_g(x)\beta_g(y)} - \overline{xy}, \gamma_g(z)\rangle = 0$$

for any $z \in \mathbb{O}_{\mathbb{Z}}$. The bilinear form $\langle\ ,\ \rangle$ is non-degenerate, so $\overline{\alpha_g(x)\beta_g(y)} = \gamma_g(\overline{xy})$ holds for any $x, y \in \mathbb{O}_{\mathbb{Z}}$. By [Yok09, Lemma 1.14.4], we have $\alpha_g, \beta_g, \gamma_g \in \mathrm{SO}(\mathbb{O}_{\mathbb{Z}})$.

Now we have obtained an injective homomorphism $\mathscr{D} \to \widetilde{\mathrm{SO}(\mathbb{O}_{\mathbb{Z}})}$. Conversely, by the definition of the multiplication $\circ$ and the condition on $(\alpha, \beta, \gamma) \in \widetilde{\mathrm{SO}(\mathbb{O}_{\mathbb{Z}})}$, the morphism

$$[a, b, c\,; x, y, z] \mapsto [a, b, c\,; \alpha(x), \beta(y), \gamma(z)]$$

lies in $\mathscr{D}$, thus $\mathscr{D} \simeq \widetilde{\mathrm{SO}(\mathbb{O}_{\mathbb{Z}})}$. □

Let $\varphi : \widetilde{\mathrm{SO}(\mathbb{O}_{\mathbb{Z}})} \to \mathrm{SO}(\mathbb{O}_{\mathbb{Z}})$ be the homomorphism sending a triple $(\alpha, \beta, \gamma) \in \widetilde{\mathrm{SO}(\mathbb{O}_{\mathbb{Z}})}$ to its third entry $\gamma \in \mathrm{SO}(\mathbb{O}_{\mathbb{Z}})$.

Proof of Proposition 2.3.9. For the bound on $|\mathcal{F}_{4,\mathrm{I}}(\mathbb{Z})|$, it suffices to prove

$$|\widetilde{\mathrm{SO}(\mathbb{O}_{\mathbb{Z}})}| \leq 2^{14} \cdot 3^5 \cdot 5^2 \cdot 7.$$

Let $(\alpha, \beta, \mathrm{id})$ be an element in $\ker\varphi$, so $\alpha(x)\beta(y) = xy$ for all $x, y \in \mathbb{O}_{\mathbb{Z}}$. Set $r = \beta(1)$ and we have $\alpha(x) = xr^{-1}$ and $\beta(y) = ry$. Setting $z = xr^{-1}$, the relation satisfied by $(\alpha, \beta, \mathrm{id})$ becomes:

$$z(ry) = (zr)y, \text{ for all } y, z \in \mathbb{O}_{\mathbb{Z}}.$$

According to [CS03, §8, Theorem 1], the octonion r of norm 1 is real, thus $r = \pm 1$ and $\ker\varphi = \{(\mathrm{id}, \mathrm{id}, \mathrm{id}), (-\mathrm{id}, -\mathrm{id}, \mathrm{id})\}$. As a consequence, we have

$$|\widetilde{\mathrm{SO}(\mathbb{O}_{\mathbb{Z}})}| \leq 2 \cdot |\,\mathrm{SO}(\mathbb{O}_{\mathbb{Z}})| = |\,\mathrm{O}(\mathbb{O}_{\mathbb{Z}})| = |\mathrm{W}(\mathrm{E}_8)| = 2^{14} \cdot 3^5 \cdot 5^2 \cdot 7,$$

which gives us the desired upper bound for $|\mathcal{F}_{4,\mathrm{I}}(\mathbb{Z})|$.

Suppose that the reductive $\mathbb{Z}$-model $\mathcal{F}_{4,\mathrm{I}}$ of $\mathbf{F}_4$ is $\mathbf{F}_4(\mathbb{Q})$-conjugate to $\mathcal{F}_{4,\mathrm{E}}$, then their $\mathbb{Z}$-points have the same order as finite groups. In the end of §2.3.2, we prove that $|\mathcal{F}_{4,\mathrm{E}}(\mathbb{Z})| = 2^{12} \cdot 3^5 \cdot 7^2 \cdot 13$, thus with the same order, the group $\mathcal{F}_{4,\mathrm{I}}(\mathbb{Z})$ contains an element of order 13. However, $\mathcal{F}_{4,\mathrm{I}}(\mathbb{Z})$ is isomorphic to $\widetilde{\mathrm{SO}(\mathbb{O}_{\mathbb{Z}})} \rtimes \mathrm{S}_3$, whose order is not divided by 13. This leads to a contradiction. □

Now Proposition 2.3.6 and Proposition 2.3.9 together imply Proposition 2.3.5, and as a corollary the equality in the upper bound in Proposition 2.3.9 holds:

Corollary 2.3.13. *The finite group $\mathcal{F}_{4,\mathrm{I}}(\mathbb{Z})$ has order $2^{15} \cdot 3^6 \cdot 5^2 \cdot 7$, and φ is surjective.*

3 Dimensions of spaces of invariants for F_4

For a finite subgroup Γ and an irreducible representation U of the compact Lie group F_4, an interesting problem is to compute the dimension of the space of invariants U^Γ. In this section we will give an algorithm to compute $\dim U^\Gamma$ for $\Gamma = \mathcal{F}_{4,\mathrm{I}}(\mathbb{Z})$ or $\mathcal{F}_{4,\mathrm{E}}(\mathbb{Z})$. These dimensions will play an important role in our computation of spaces of automorphic forms in §5.1.1. The code of the computations in this section can be found in [Sha].

3.1 Ideas and obstructions

By the highest weight theory, the isomorphism classes of irreducible $\mathbb{C}$-representations of the compact Lie group F_4 are in natural bijection with dominant weights of the irreducible root system F_4. Using notations in [Bou07, §IV.4.9], we denote the weight $\lambda_1\varpi_1 + \lambda_2\varpi_2 + \lambda_3\varpi_3 + \lambda_4\varpi_4$ by $\lambda = (\lambda_1, \lambda_2, \lambda_3, \lambda_4)$, where $\varpi_1, \varpi_2, \varpi_3, \varpi_4$ are the four fundamental weights of F_4. Let V_λ be a representative of the isomorphism class of irreducible representations of F_4 with highest weight λ. From now on we call V_λ *the* irreducible representation of F_4 with highest weight λ for short.

The starting point of the computation of $\dim \mathrm{V}_\lambda^\Gamma$ for some finite subgroup Γ of F_4 is the following classic lemma:

Lemma 3.1.1. *For a finite subgroup $\Gamma \subset \mathrm{F}_4$, we have*

$$\dim \mathrm{V}_\lambda^\Gamma = \frac{1}{|\Gamma|}\sum_{\gamma\in\Gamma} \mathrm{Tr}|_{\mathrm{V}_\lambda}(\gamma) = \frac{1}{|\Gamma|}\sum_{c\in\mathrm{Conj}(\Gamma)} \mathrm{Tr}|_{\mathrm{V}_\lambda}(c)\cdot|c|,$$

where $\mathrm{Conj}(\Gamma)$ is the set of conjugacy classes of Γ and $|c|$ denotes the cardinality of c.

Because of this lemma, it is enough to solve the following two problems to compute $\dim \mathrm{V}_\lambda^\Gamma$:

(i) Find all conjugacy classes of Γ, and choose a representative in a fixed maximal torus $T \subset \mathrm{F}_4$ for each conjugacy class;

(ii) For an element $t \in T$, compute its trace $\mathrm{Tr}|_{\mathrm{V}_\lambda}(t)$.

Problem (ii) can be dealt with the following *degenerate Weyl character formula*:

Proposition 3.1.2. *[CR15, Proposition 2.1] Let G be a connected compact Lie group, T a maximal torus, $X = \mathrm{X}^*(T)$ the character group of T, and Φ the root system of (G,T) with Weyl group W. Choose a system of positive roots $\Phi^+ \subset \Phi$ with base Δ and also fix a W-invariant inner product $(\ ,\ )$ on $X \otimes_\mathbb{Z} \mathbb{R}$. Let λ be a dominant weight in X and t an element in T. Denote the connected component $\mathrm{C}_G(t)^\circ$ of the centralizer of t by M. Set $\Phi_M^+ = \Phi(M,T)\cap\Phi^+$ and $W^M = \{w\in W : w^{-1}\Phi_M^+ \subset \Phi^+\}$. Let ρ and ρ_M be the half-sum of the elements of Φ^+ and Φ_M^+ respectively. We have:*

$$\mathrm{Tr}|_{\mathrm{V}_\lambda}(t) = \frac{\sum_{w\in W^M}\varepsilon(w)t^{w(\lambda+\rho)-\rho}\cdot\prod_{\alpha\in\Phi_M^+}\frac{(\alpha,w(\lambda+\rho))}{(\alpha,\rho_M)}}{\prod_{\alpha\in\Phi^+\setminus\Phi_M^+}(1-t^{-\alpha})}, \tag{3.1}$$

where $\varepsilon : W \to \{\pm1\}$ is the signature and t^x denotes $x(t)$ for convenience.

Using this approach, problem (i) is thus the main difficulty for our computation, and we will solve it in the following subsections.

3.2 Generators of $\mathcal{F}_{4,\mathrm{I}}(\mathbb{Z})$ and $\mathcal{F}_{4,\mathrm{E}}(\mathbb{Z})$

The finite groups Γ we are interested in are $\mathcal{F}_{4,\mathrm{I}}(\mathbb{Z})$ and $\mathcal{F}_{4,\mathrm{E}}(\mathbb{Z})$. To find all their conjugacy classes, we first determine generators of these groups in this subsection.

In the end of §2.3.2, we have already showed that the group $\mathcal{F}_{4,\mathrm{E}}(\mathbb{Z})$ is generated by two elements σ_1, σ_2. Their matrices in the basis $\mathcal{B}$, given in (2.6), are written down in Figure 2, Appendix A.

Based on Corollary 2.3.13, we have $\mathcal{F}_{4,\mathrm{I}}(\mathbb{Z}) = \mathscr{D} \rtimes \kappa(\mathrm{S}_3)$, where $\kappa : \mathrm{S}_3 \to \mathcal{F}_{4,\mathrm{I}}(\mathbb{Z})$ is the morphism defined in (2.7). The group $\mathscr{D}$ is isomorphic to the group $\widetilde{\mathrm{SO}(\mathbb{O}_\mathbb{Z})}$, which is a double cover of $\mathrm{SO}(\mathbb{O}_\mathbb{Z})$ by Corollary 2.3.13. Therefore it suffices to find generators of $\mathscr{D}$.

Since $\mathrm{O}(\mathbb{O}_\mathbb{Z}) \simeq \mathrm{O}(\mathrm{E}_8)$ is equal to the Weyl group of E_8, we can take the following set of generators for $\mathrm{SO}(\mathbb{O}_\mathbb{Z})$:

$$\{\mathrm{ref}(\alpha) \circ \mathrm{ref}(1) \mid \alpha \in \mathbb{O}_\mathbb{Z}, \mathrm{N}(\alpha) = 1\},$$

where for a root α in $\mathbb{O}_\mathbb{Z}$ the reflection $\mathrm{ref}(\alpha)$ is defined as

$$\mathrm{ref}(\alpha)(x) := x - \langle x, \alpha\rangle\alpha.$$

For a root $\alpha \in \mathbb{O}_\mathbb{Z}$, let L_α (resp. R_α) be the left (resp. right) multiplication on $\mathbb{O}_\mathbb{Z}$ by α, and define $\mathrm{B}_\alpha := \mathrm{L}_\alpha \circ \mathrm{R}_\alpha = \mathrm{R}_\alpha \circ \mathrm{L}_\alpha$. These elements are contained in $\mathrm{SO}(\mathbb{O}_\mathbb{Z})$. Notice that for a root $\alpha \in \mathbb{O}_\mathbb{Z}$, $\mathrm{ref}(\alpha) \circ \mathrm{ref}(1) = \mathrm{B}_\alpha$.

Lemma 3.2.1. *For any root $\alpha \in \mathbb{O}_\mathbb{Z}$, the triple $(\mathrm{L}_{\overline{\alpha}}, \mathrm{R}_{\overline{\alpha}}, \mathrm{B}_\alpha)$ is an element in $\widetilde{\mathrm{SO}(\mathbb{O}_\mathbb{Z})}$.*

Proof. For any $x, y \in \mathbb{O}_\mathbb{Z}$, $\overline{\mathrm{L}_{\overline{\alpha}}(x)\mathrm{R}_{\overline{\alpha}}(y)} = \overline{(\overline{\alpha}x)(y\overline{\alpha})}$. By *Moufang laws* [CS03, §6.5],

$$(\overline{\alpha}x)(y\overline{\alpha}) = (\overline{\alpha}(xy))\overline{\alpha} = \mathrm{B}_{\overline{\alpha}}(xy),$$

thus $\overline{\mathrm{L}_{\overline{\alpha}}(x)\mathrm{R}_{\overline{\alpha}}(y)} = \overline{\mathrm{B}_{\overline{\alpha}}(xy)} = \mathrm{B}_\alpha(\overline{xy})$. □

By this lemma, we can take

$$\{(\mathrm{L}_{\overline{\alpha}}, \mathrm{R}_{\overline{\alpha}}, \mathrm{B}_\alpha) \mid \alpha \in \mathbb{O}_\mathbb{Z}, \mathrm{N}(\alpha) = 1\} \cup \{(-\mathrm{id}, -\mathrm{id}, \mathrm{id})\}$$

as generators of $\mathscr{D}$. Together with a set of generators of $\kappa(\mathrm{S}_3)$ we have obtained generators of $\mathcal{F}_{4,\mathrm{I}}(\mathbb{Z})$.

3.3 Enumeration of conjugacy classes

Now with generators of $\mathcal{F}_I(\mathbb{Z})$ and $\mathcal{F}_{4,\mathrm{E}}(\mathbb{Z})$, we can start to enumerate their conjugacy classes. The `ConjugationClasses` function in [GAP] can assist us in enumerating the conjugacy classes of subgroups of permutation groups. Therefore it is enough to realize these two finite groups as permutation groups.

For $\mathcal{F}_{4,\mathrm{I}}(\mathbb{Z})$, we consider its action on the set of vectors $v \in \mathbb{O}_\mathbb{Z}$ with $\mathrm{B}_\mathrm{Q}(v, v) \leq 2$. The function `qfminim` in [PARI/GP] can list all these vectors in the basis $\mathcal{B}$. There are 738 such vectors and they span the vector space $\mathrm{J}_\mathbb{R}$, so the action of $\mathcal{F}_{4,\mathrm{I}}(\mathbb{Z})$ on this set is faithful, which gives us an embedding $\mathcal{F}_{4,\mathrm{I}}(\mathbb{Z}) \hookrightarrow \mathrm{S}_{738}$. We can thus use this embedding to obtain a set of representatives of conjugacy classes of $\mathcal{F}_{4,\mathrm{I}}(\mathbb{Z})$ via the help of [GAP].

For the other group $\mathcal{F}_{4,\mathrm{E}}(\mathbb{Z})$ we use a similar strategy. As mentioned in Remark 2.3.8, the quadratic lattice $(\mathrm{J}_{\mathbb{Z}}, \langle\ ,\ \rangle_{\mathrm{E}})$ has no roots, so we consider the set of $v \in \mathrm{J}_{\mathbb{Z}}$ such that $\langle v, v\rangle_{\mathrm{E}} = 3$, which has cardinality 1640 and generates $\mathrm{J}_{\mathbb{R}}$. This gives an embedding $\mathcal{F}_{4,\mathrm{E}}(\mathbb{Z}) \hookrightarrow \mathrm{S}_{1640}$, then we can use [GAP].

Here we present the results, and all the codes are available in [Sha].

Proposition 3.3.1. *There are* 113 *conjugacy classes in* $\mathcal{F}_{4,\mathrm{I}}(\mathbb{Z})$*, while* $\mathcal{F}_{4,\mathrm{E}}(\mathbb{Z})$ *has* 49 *conjugacy classes.*

Furthermore, [GAP] gives the size of each conjugacy class c, and selects a representative for c in the form of permutation. We rewrite these representatives as matrices in the basis $\mathcal{B}$.

3.4 Kac coordinates

In the previous subsection, for $\Gamma = \mathcal{F}_{4,\mathrm{I}}(\mathbb{Z})$ or $\mathcal{F}_{4,\mathrm{E}}(\mathbb{Z})$, we obtained a list of its conjugacy classes and a representative element $g_c \in \Gamma$ for each conjugacy class c.

However, the representative g_c may not be contained in the fixed maximal torus in Proposition 3.1.2. Notice that in the computation of the trace of g_c for a Γ-conjugacy class c, what really matters is the F_4-conjugacy class containing c. Furthermore, since c is included in the finite group Γ, the F_4-conjugacy class containing it must be torsion.

In [Ree10], it is shown that we can choose a representative for a torsion F_4-conjugacy class in a fixed maximal torus using its *Kac coordinates*. Here we provide a brief review, and more details can be found in Reeder's paper.

Let G be a simply-connected simple compact Lie group, T a fixed maximal torus, $X := \mathrm{X}^*(T)$ and $Y := \mathrm{X}_*(T)$ the groups of characters and cocharacters respectively, and Φ the root system of (G, T). Denote the natural pairing $X \times Y \to \mathbb{Z}$ by $\langle\ ,\ \rangle$. Let $\Delta = \{\alpha_1, \ldots, \alpha_r\}$ be a set of simple roots of Φ, and $\{\check{\varpi}_1, \ldots, \check{\varpi}_r\}$ its dual basis in Y, i.e. $\langle \alpha_i, \check{\varpi}_j\rangle = \delta_{ij}$.

We have a surjective *exponential map* $\exp : Y \otimes_{\mathbb{Z}} \mathbb{R} \to T$ determined uniquely by the property

$$\alpha\left(\exp(y)\right) = e^{2\pi i\langle \alpha, y\rangle}, \forall \alpha \in X, y \in Y \otimes_{\mathbb{Z}} \mathbb{R}.$$

and Y is the kernel of this exponential map. This induces an isomorphism $(Y \otimes_{\mathbb{Z}} \mathbb{R})/Y \simeq T$.

Let $\widetilde{\alpha}_0 = \sum_{i=1}^{r} a_i\alpha_i$ be the highest root with respect to the choice of simple roots Δ, and set $\alpha_0 = 1 - \widetilde{\alpha}_0, a_0 = 1$ and $\check{\varpi}_0 = 0$. Now we have $\sum_{i=0}^{r} a_i\alpha_i = 1$. The *alcove* determined by Δ is the intersection of half-spaces:

$$C = \{x \in Y \otimes_{\mathbb{Z}} \mathbb{R} \,|\, \langle \alpha_i, x\rangle > 0, \forall i = 0, 1, \ldots, r\},$$

or

$$\overline{C} = \left\{ \sum_{i=0}^{r} x_i\check{\varpi}_i \,\middle|\, \sum_{i=0}^{r} a_i x_i = 1, x_i \geq 0, \forall i = 0, 1, \ldots, r \right\}.$$

Each torsion element $s \in G$ is conjugate to $\exp(x)$ for a unique $x \in \overline{C} \cap (Y \otimes_{\mathbb{Z}} \mathbb{Q})$ since the group G is simply-connected. Let m be the order of s, thus

$$x = \frac{1}{m} \sum_{i=1}^{r} s_i \check{\varpi}_i$$

for some non-negative integers $s_1, \ldots, s_r$ satisfying $\gcd\{m, s_1, \ldots, s_r\} = 1$.

Since $x \in \overline{C}$, we set $s_0 := m - \sum_{i=1}^{r} a_i s_i \geq 0$. Now the non-negative integers $s_0, s_1, \ldots, s_r$ satisfy $\gcd\{s_0, \ldots, s_r\} = 1$ and the equation

$$\sum_{i=0}^{r} a_i s_i = m \text{ with } a_0 = 1.$$

The coordinates $(s_0, s_1, \ldots, s_r)$ are called the *Kac coordinates of s*, which are uniquely determined by the G-conjugacy class of s.

In our case, the compact group F_4 is simply-connected and the highest root $\widetilde{\alpha}_0 = 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4$. Here $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ are still chosen as in [Bou07, §IV.4]. In conclusion, we have:

Proposition 3.4.1. *Let T be a fixed maximal torus of F_4. Any element of order m in F_4 is conjugate to a unique element $\exp(\frac{\sum_{i=1}^{4} s_i \varpi_i}{m})$ for some non-negative integers s_1, s_2, s_3, s_4 arising from a 5-tuple $(s_0, s_1, s_2, s_3, s_4)$ in*

$$\left\{(x_0, \ldots, x_4) \in \mathbb{N}^5 \,\middle|\, x_0 + 2x_1 + 3x_2 + 4x_3 + 2x_4 = m, \gcd\{x_0, \ldots, x_4\} = 1\right\}. \tag{3.2}$$

By solving the equation in (3.2), we enumerate all the torsion F_4-conjugacy classes of order m.

3.5 Comparison of conjugacy classes

Now we can enumerate F_4-conjugacy classes of a given order, but there are more constraints on the F_4-conjugacy classes containing Γ-conjugacy classes obtained in §3.3. So we define the following class of F_4-conjugacy classes:

Definition 3.5.1. Let c be an F_4-conjugacy class, and we say that c is a *rational conjugacy class* if it satisfies:

- its trace $\mathrm{Tr}(c)|_{\mathfrak{f}_4}$ on the adjoint representation $\mathfrak{f}_4$ of F_4 is a rational number;
- its characteristic polynomial $\mathrm{P}_c(X) := \det(X \cdot \mathrm{id} - g|_{\mathrm{J}_{\mathbb{C}}})$ on $\mathrm{J}_{\mathbb{C}} := \mathrm{J}_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$, $g \in \mathrm{F}_4$ being a representative of c, has rational coefficients.

For $\Gamma = \mathcal{F}_{4,\mathrm{I}}(\mathbb{Z})$ or $\mathcal{F}_{4,\mathrm{E}}(\mathbb{Z})$, since Γ is a subgroup of $\mathrm{GL}(\mathrm{J}_{\mathbb{Z}})$, the F_4-conjugacy class containing a Γ-conjugacy class of Γ must be rational in the sense of Definition 3.5.1.

Our strategy in this subsection is:

(1) find all rational torsion F_4-conjugacy classes, and for each of them choose a representative in the maximal torus T fixed before in §3.4;

(2) determine which F_4-conjugacy class contains a given Γ-conjugacy class by comparing their traces and characteristic polynomials.

Before explaining the algorithm for step (1), we state the following lemma:

Lemma 3.5.2. *If m is the order of an element in F_4 whose characteristic polynomial on $\mathrm{J}_{\mathbb{C}}$ has rational coefficients, then $m = 66, 70, 72, 78, 84$ or 90, or $m \leq 60$.*

Proof. As a representation of F_4, $\mathrm{J}_{\mathbb{C}}$ is isomorphic to $\mathrm{V}_{\varpi_4} \oplus \mathbb{C}$, where $\mathbb{C}$ stands for the trivial representation. Since the zero weight appears twice in the weights of V_{ϖ_4}, the characteristic polynomial is divisible by $(X-1)^3$. On the other hand, the roots of this polynomial contain a primitive mth root of unity, thus the polynomial is also divisible by the mth cyclotomic polynomial. Hence we have $\varphi(m) \leq 24$, where φ denotes the Euler function. This implies $m \leq 60$, or $m = 66, 70, 72, 78, 84$ or 90.

□

With the help of [PARI/GP], we enumerate all the Kac coordinates $s = (s_0, s_1, s_2, s_3, s_4)$ satisfying the conditions in (3.2) for each integer m in

$$\{n \leq 60 \,|\, \varphi(n) \leq 24\} \cup \{66, 70, 72, 78, 84, 90\}.$$

For each such s, we compute the trace on $\mathfrak{f}_4$ and the characteristic polynomial on $\mathrm{J}_{\mathbb{C}}$ of the corresponding element $t = \exp(\frac{\sum_{i=1}^4 s_i \varpi_i}{m}) \in T$. Using this algorithm, we get the Kac coordinates of all rational torsion F_4-conjugacy classes.

Proposition 3.5.3. *There are exactly 102 rational torsion conjugacy classes in F_4, whose Kac coordinates are listed in Table 4.*

Our result coincides with [Pad98, Table 9.1]. In Table 4, we also list the invariants defined below for all rational torsion F_4-conjugacy class.

For a representative $g \in \mathrm{F}_4$ of a rational torsion conjugacy class c, we can compute its characteristic polynomial on $\mathrm{J}_{\mathbb{C}}$:

$$\mathrm{P}_g(X) = \det\left(X \cdot \mathrm{id} - g|_{\mathrm{J}_{\mathbb{C}}}\right) = \sum_{i=0}^{27} (-1)^{i+1} a_i(g) X^i.$$

Now we assign to g a quadruple

$$\mathrm{i}(g) := \left(a_{26}(g), a_{25}(g), a_{24}(g), \mathrm{Tr}(\mathrm{Ad}(g)|_{\mathfrak{f}_4})\right),$$

and set $\mathrm{i}(c) := \mathrm{i}(g)$.

Corollary 3.5.4. *Let g_1, g_2 be two elements in either $\mathcal{F}_{4,\mathrm{I}}(\mathbb{Z})$ or $\mathcal{F}_{4,\mathrm{E}}(\mathbb{Z})$, then g_1 and g_2 are conjugate in F_4 if and only if $\mathrm{i}(g_1) = \mathrm{i}(g_2)$.*

Proof. This follows from Table 4. For each rational torsion conjugacy class c, we list its order $o(c)$ and the associated quadruple $i(c)$. We observe that two different classes c have different $i(c)$. □

Remark 3.5.5. There exist examples of two different rational torsion conjugacy classes in F_4 whose characteristic polynomials on $\mathrm{J}_{\mathbb{C}}$ are the same. For instance, the order 12 conjugacy classes c_1 and c_2 represented by the Kac coordinates $(1,1,1,1,1)$ and $(2,1,0,1,2)$ respectively share the same characteristic polynomial on $\mathrm{J}_{\mathbb{C}}$:

$$X^{27} - X^{24} - 2X^{15} + 2X^{12} + X^3 - 1.$$

However, the trace of c_1 on $\mathfrak{f}_4$ is 0, while that of c_2 is 3. This shows that the 26-dimensional irreducible representation of F_4 is not "excellent" in the sense of Padowitz. It is also observed in Padowitz's table [Pad98, Table 9.1] that the motives attached to the centralizers of these two conjugacy classes, in the sense of Gross, are different.

Now we explain our algorithm for step (2). For each Γ-conjugacy class c and its representative g_c chosen in §3.3, we compute the quadruple $\mathrm{i}(g_c)$ and compare it with Table 4. By Corollary 3.5.4 we can determine the F_4-conjugacy class containing c. In Table 5 we list all the Kac coordinates s whose corresponding rational conjugacy class c_s in F_4 satisfies that $c_s \cap \mathcal{F}_{4,\mathrm{I}}(\mathbb{Z})$ or $c_s \cap \mathcal{F}_{4,\mathrm{E}}(\mathbb{Z})$ is non-empty, as well as the cardinalities of intersections $n_1(s) = |c_s \cap \mathcal{F}_{4,\mathrm{I}}(\mathbb{Z})|$ and $n_2(s) = |c_s \cap \mathcal{F}_{4,\mathrm{E}}(\mathbb{Z})|$.

3.6 The formula for $\dim \mathrm{V}_\lambda^\Gamma$

Now we can deduce the formula for $d_i(\lambda) := \dim \mathrm{V}_\lambda^{\Gamma_i}, i = 1,2$, where $\Gamma_1 := \mathcal{F}_{4,\mathrm{I}}(\mathbb{Z})$ and $\Gamma_2 := \mathcal{F}_{4,\mathrm{E}}(\mathbb{Z})$, for a given dominant weight λ:

$$\dim \mathrm{V}_\lambda^{\Gamma_i} = \frac{1}{|\Gamma_i|} \sum_{c \in \mathrm{Conj}(\Gamma_i)} \mathrm{Tr}|_{\mathrm{V}_\lambda}(c) \cdot |c| = \frac{1}{|\Gamma_i|} \sum_{c \in \mathrm{Conj}(\mathrm{F}_4)} \mathrm{Tr}|_{\mathrm{V}_\lambda}(c) \cdot |c \cap \Gamma_i|.$$

For each rational conjugacy class c whose contribution to this formula is nonzero, we have already given $|c \cap \Gamma_i|$ in Table 5, and according to Proposition 3.1.2 the trace $\mathrm{Tr}|_{\mathrm{V}_\lambda}(c')$ is an explicit function of $\lambda_1, \lambda_2, \lambda_3, \lambda_4$.

This gives us the following theorem, which is the main computational result of this paper:

Theorem 3.6.1. *For each dominant weight λ of the compact Lie group F_4, we have an explicit formula for*

$$d_i(\lambda) = \dim \mathrm{V}_\lambda^{\Gamma_i}, i = 1,2.$$

For dominant weights $\lambda = (\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ with $2\lambda_1 + 3\lambda_2 + 2\lambda_3 + \lambda_4 \leq 13$, we list all the nonzero $d(\lambda) := d_1(\lambda) + d_2(\lambda)$ in Table 6.

Remark 3.6.2. Later we will see the condition on λ in Theorem 3.6.1 is equivalent to that the maximal eigenvalue of the infinitesimal character associated to V_λ is not larger than 21.

In [Sha], we also provide a larger table of $[\lambda, d_1(\lambda), d_2(\lambda), d(\lambda)]$ for weights with $2\lambda_1 + 3\lambda_2 + 2\lambda_3 + \lambda_4 \leq 40$.

4 Subgroups of F_4

In this section we will classify subgroups of the compact Lie group $\mathrm{F}_4 = \mathrm{Aut}(\mathrm{J}_{\mathbb{R}}, \circ)$ satisfying certain conditions and determine their centralizers in F_4. Our results will be used in §6, but this problem also has its own interest. Our precise aim is to find all the conjugacy classes of closed subgroups H of F_4 such that:

(1) H is connected;
(2) The centralizer of H in F_4 is an elementary finite abelian 2-groups, i.e. it is a product of finitely many copies of $\mathbb{Z}/2\mathbb{Z}$.
(3) The multiplicity of zero weight in the restriction of the 26-dimensional irreducible representation V_{ϖ_4} of F_4 to H is 2.

If we only consider the first condition, the problem is equivalent to classifying connected semisimple Lie subalgebras of the complexified Lie algebra $\mathfrak{f}_4$, up to the adjoint action of $\mathbf{F}_4(\mathbb{C})$. This has been studied by Dynkin in [Dyn57] for all simple complex Lie algebras, without giving full details. So we will give a detailed classification for F_4 in this section, following Dynkin's original idea and Losev's result [Los10, Theorem 7.1].

Briefly, our strategy is to enumerate first all the connected simple subgroups of F_4 inside maximal proper compact subgroups, and to index them by the restrictions of V_{ϖ_4}. Then we compute their centralizers case by case, and combine these results together to get all the connected subgroups satisfying our conditions.

4.1 Element-conjugacy implies conjugacy

To be more precise, what we want to classify, up to F_4-conjugacy, are embeddings from connected compact Lie groups to F_4 satisfying two additional conditions. In this subsection we will explain why it is enough to consider their element-conjugacy classes, where the notion of element-conjugacy is defined as follows:

Definition 4.1.1. [FHS16, §1] Let G and H be two compact Lie groups and $\phi, \phi' : H \to G$ be two Lie group homomorphisms. We say that ϕ and ϕ' are *conjugate* if there is an element $g \in G$ such that
$$g\phi(h)g^{-1} = \phi'(h), \text{ for all } h \in H.$$
They are said to be *element-conjugate* if for every $h \in H$, there is a $g \in G$ such that
$$g\phi(h)g^{-1} = \phi'(h).$$

The element-conjugacy can be rephrased in the following way:

Lemma 4.1.2. *Let $\phi, \phi' : H \to G$ be two homomorphisms between compact Lie groups, then they are element-conjugate if and only if for each linear representation $\pi : G \to \mathrm{GL}(V)$ the compositions $\pi \circ \phi$ and $\pi \circ \phi'$ are conjugate in $\mathrm{GL}(V)$.*

Proof. It is a consequence of the Peter-Weyl theorem for compact Lie groups, which says that two elements of G are conjugate if and only if they have the same trace on all the irreducible representations of G. $\square$

It is obvious that two conjugate homomorphisms are element-conjugate, but the converse fails in general. Fortunately, the converse holds when $G = \mathrm{F}_4$, due to the following result for Lie algebras:

Theorem 4.1.3. *[Los10, Proposition 6.2, Theorem 7.1] Let $\mathfrak{f}_4$ be a simple complex Lie algebra of type F_4 and $\mathrm{F}_{4,\mathbb{C}}$ the complexification of F_4. Let $\mathfrak{h}$ be a reductive algebraic Lie algebra, i.e. $\mathfrak{h}$ is the Lie algebra of some reductive complex group, and $\phi, \phi' : \mathfrak{h} \to \mathfrak{f}_4$ two injective Lie algebra homomorphisms. If the restrictions of ϕ and ϕ' to a Cartan subalgebra $\mathfrak{s}$ of $\mathfrak{h}$ are conjugate in the sense that $\varphi \circ \phi|_{\mathfrak{s}} = \phi'|_{\mathfrak{s}}$ for an inner automorphism φ of $\mathfrak{f}_4$, then ϕ and ϕ' are conjugate.*

Remark 4.1.4. Actually, in [Los10] Losev uses the following equivalence relation on Lie algebra homomorphisms: two Lie algebra homomorphisms $\phi, \phi' : \mathfrak{h} \to \mathfrak{g}$ are equivalent if there exist liftings $H \to G$ of ϕ, ϕ' to reductive complex groups which are G-conjugate in the sense of Definition 4.1.1. By Lie group-Lie algebra correspondence this equivalence relation is the same as $\varphi \circ \phi = \phi'$ for an inner automorphism φ of $\mathfrak{f}_4$.

This theorem implies the result we need for F_4:

Proposition 4.1.5. *For any connected compact Lie group H, two element-conjugate homomorphisms from H to F_4 are conjugate.*

Proof. The argument that deduces this result from Theorem 4.1.3 can be found in the proof of [FHS16, Proposition 3.5]. □

4.2 A criterion for element-conjugacy

According to Lemma 4.1.2 and Proposition 4.1.5, to check whether two homomorphism ϕ and ϕ' from a connected compact Lie group H to F_4 are conjugate, it suffices to verify that for every irreducible representation π of F_4, $\pi \circ \phi$ and $\pi \circ \phi'$ are equivalent as H-representations. Moreover, we have the following useful fact:

Proposition 4.2.1. *Let (π_0, J_0) be the 26-dimensional irreducible representation of F_4. Two homomorphisms ϕ, ϕ' from a connected compact subgroup H to F_4 are conjugate if and only if two H-representations $\pi_0 \circ \phi$ and $\pi_0 \circ \phi'$ are equivalent.*

This result is a part of [Dyn57, Theorem 1.3], but Dynkin only gives a short sketch of the proof, so in this subsection we will give the proof of Proposition 4.2.1.

We first give a preliminary discussion on orders. Let X be an abelian group and $\ell : X \to \mathbb{R}$ a $\mathbb{Z}$-linear map. This map induces a total preorder $\leq$ on X defined by $x \leq y$ if and only if $\ell(x) \leq \ell(y)$. A preorder on X of this form will be called an *L-preorder*. If the map ℓ is injective, the L-preorder it induces is an order and we call this order an *L-order*. For instance, any free abelian group of finite rank admits L-orders.

Lemma 4.2.2. *Let $f : X \to Y$ be a homomorphism between finitely generated free abelian groups X and Y, with an L-order on Y, and S a finite subset of $X - \{0\}$. There exists an L-preorder $\leq$ on X such that for any $s \in S$ we have either $s > 0$ or $s < 0$, and if $s > 0$ then $f(s) \geq 0$ in Y.*

Proof. We choose $\ell : Y \hookrightarrow \mathbb{R}$ such that the L-order on Y is defined by ℓ. Write $S = S_0 \sqcup S_1$, with $S_0 = S \cap \ker f$. If S_0 is empty, then the L-preorder on X defined by $\ell \circ f$ satisfies the conditions.

If S_0 is not empty, we choose an arbitrary injective $\mathbb{Z}$-linear map $j : X \hookrightarrow \mathbb{R}$ and set

$$\varepsilon := \frac{1}{2} \min_{s \in S_1} \frac{|\ell(f(s))|}{|j(s)|}.$$

We claim that the L-preorder on X defined by $j' = \ell \circ f + \varepsilon j$ satisfies the desired conditions. Indeed, for $s \in S_0$, $j'(s) = \varepsilon j(s)$ is nonzero. Also for $s \in S_1$, by our choice of ε, we have $|\varepsilon j(s)| < |\ell(f(s))|$, so $j'(s)$ is nonzero and of the same sign as $\ell(f(s))$. $\square$

The next lemma concerns the partial order $\preceq$ of the weights of the 26-dimensional irreducible representation π_0 of F_4. Recall that for two weights λ and μ of F_4, fixing a positive root system of F_4, we write $\lambda \succeq \mu$ if $\lambda - \mu$ is a finite sum of positive roots.

Lemma 4.2.3. *The* 26*-dimensional irreducible representation* (π_0, J_0) *of* F_4 *has four unique weights* $\lambda_1 \succ \lambda_2 \succ \lambda_3 \succ \lambda_4$ *satisfying that* $\lambda \prec \lambda_4$ *for all other weights* λ*. Moreover, those* 4 *weights* $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ *form a* $\mathbb{Z}$*-basis of the weight lattice of* F_4*.*

Proof. Fix a maximal torus T of F_4, and let $X = \mathrm{X}^*(T)$ be its character lattice and $\Phi^+ \subset X$ a positive root system with respect to (F_4, T). We still use Bourbaki's notations [Bou07, §IV.4.9] for the root system F_4. The simple roots with respect to Φ^+ are given by

$$\alpha_1 = \varepsilon_2 - \varepsilon_3, \alpha_2 = \varepsilon_3 - \varepsilon_4, \alpha_3 = \varepsilon_4, \alpha_4 = \frac{1}{2}(\varepsilon_1 - \varepsilon_2 - \varepsilon_3 - \varepsilon_4),$$

where $\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4$ is the basis of $X \otimes_{\mathbb{Z}} \mathbb{R} \simeq \mathbb{R}^4$ chosen in [Bou07] satisfying

$$X = \mathbb{Z}\varepsilon_1 + \mathbb{Z}\varepsilon_2 + \mathbb{Z}\varepsilon_3 + \mathbb{Z}\varepsilon_4 + \mathbb{Z}\frac{\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4}{2}.$$

The highest weight of π_0 is $\varpi_4 = \alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4 = \varepsilon_1$. The orbit of ϖ_4 under the Weyl group consists of $\pm\varepsilon_i$ for $i = 1, 2, 3, 4$ and $\frac{1}{2}(\pm\varepsilon_1 \pm \varepsilon_2 \pm \varepsilon_3 \pm \varepsilon_4)$. These 24 weights have multiplicity 1, and the zero weight appears with multiplicity 2.

We claim that the weights

$$\lambda_1 = \varepsilon_1, \lambda_2 = \frac{1}{2}(\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4),$$
$$\lambda_3 = \frac{1}{2}(\varepsilon_1 + \varepsilon_2 + \varepsilon_3 - \varepsilon_4), \lambda_4 = \frac{1}{2}(\varepsilon_1 + \varepsilon_2 - \varepsilon_3 + \varepsilon_4)$$

satisfy the desired properties. Indeed, this follows from the following table:

positive weight λ	relation with $\lambda_1, \lambda_2, \lambda_3, \lambda_4$
ε_1	λ_1
ε_2	$\lambda_4 - \alpha_3 - \alpha_4$
ε_3	$\lambda_4 - \alpha_1 - \alpha_3 - \alpha_4$
ε_4	$\lambda_4 - \alpha_1 - \alpha_2 - \alpha_3 - \alpha_4$
$(\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4)/2$	$\lambda_2 = \lambda_1 - \alpha_4$
$(\varepsilon_1 + \varepsilon_2 + \varepsilon_3 - \varepsilon_4)/2$	$\lambda_3 = \lambda_2 - \alpha_3$
$(\varepsilon_1 + \varepsilon_2 - \varepsilon_3 + \varepsilon_4)/2$	$\lambda_4 = \lambda_3 - \alpha_2$
$(\varepsilon_1 + \varepsilon_2 - \varepsilon_3 - \varepsilon_4)/2$	$\lambda_4 - \alpha_3$
$(\varepsilon_1 - \varepsilon_2 + \varepsilon_3 + \varepsilon_4)/2$	$\lambda_4 - \alpha_1$
$(\varepsilon_1 - \varepsilon_2 + \varepsilon_3 - \varepsilon_4)/2$	$\lambda_4 - \alpha_1 - \alpha_3$
$(\varepsilon_1 - \varepsilon_2 - \varepsilon_3 + \varepsilon_4)/2$	$\lambda_4 - \alpha_1 - \alpha_2 - \alpha_3$
$(\varepsilon_1 - \varepsilon_2 - \varepsilon_3 - \varepsilon_4)/2$	$\lambda_4 - \alpha_1 - \alpha_2 - 2\alpha_3$

Table 1: Positive weights of the 26-dimensional irreducible representation V_{ϖ_4} of F_4

and the following identities:

$$\varepsilon_1 = \lambda_1, \varepsilon_2 = -\lambda_1 + \lambda_3 + \lambda_4, \varepsilon_3 = \lambda_2 - \lambda_4, \varepsilon_4 = \lambda_2 - \lambda_3, \frac{\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4}{2} = \lambda_2.$$

□

Proof of Proposition 4.2.1. By Proposition 4.1.5 it suffices to show that if $\pi_0 \circ \phi$ and $\pi_0 \circ \phi'$ are equivalent as H-representations, then ϕ and ϕ' are element-conjugate. Since any element of H is included in some maximal torus, we may assume that H is a torus.

We fix a maximal torus T of F_4. As all maximal tori are conjugate in F_4, up to replacing ϕ and ϕ' by some F_4-conjugate, we assume that both $\phi(H)$ and $\phi'(H)$ are contained in T. Let $X = \mathrm{X}^*(T)$ and $Y = \mathrm{X}^*(H)$, then ϕ and ϕ' induce $\mathbb{Z}$-linear maps $\phi^*, \phi^{\prime,*} : X \to Y$ respectively.

Choose an arbitrary L-order on Y, and denote by $\Phi \subset X$ the root system of (F_4, T). By Lemma 4.2.2, there is an L-preorder $\leq$ (resp. $\leq'$) on X such that for any $\alpha \in \Phi$ we have either $\alpha > 0$ or $\alpha < 0$ (resp. either $\alpha >' 0$ or $\alpha <' 0$), and the $\mathbb{Z}$-linear map ϕ^* (resp. $\phi^{\prime,*}$) preserves the preorders on X, Y. We denote the positive root system determined by the L-preorder $\leq$ (resp. $\leq'$) by Φ^+ (resp. $\Phi^{+,\prime}$).

A general fact about root systems is that the Weyl group of (F_4, T) acts transitively on the set of positive root systems of (F_4, T). Up to conjugating ϕ' by a suitable element in the normalizer $\mathrm{N}_{\mathrm{F}_4}(T)$, we may assume that $\Phi^{+,\prime} = \Phi^+$. Now our aim is to show $\phi = \phi'$, which is equivalent to $\phi^* = \phi^{\prime,*}$.

Let $\mathcal{W}$ be the multiset of X consisting of the weights appearing in π_0. Let $\lambda_1 \succ \lambda_2 \succ \lambda_3 \succ \lambda_4$ be the 4 weights of π_0 defined in Lemma 4.2.3 and all of them have multiplicity 1 in π_0. For the $\mathbb{Z}$-linear map $f = \phi^*$ or $\phi^{\prime,*}$, the preorder-preserving property of f and Table 1 imply that $f(\lambda_1) \geq f(\lambda_2) \geq f(\lambda_3) \geq f(\lambda_4)$ and $f(\lambda_4) \geq f(\lambda)$ for all other weights λ of π_0. In other words, $f(\lambda_1)$ is the greatest element of $f(\mathcal{W})$, and for $i = 2, 3, 4$, $f(\lambda_i)$ is the greatest element of $f(\mathcal{W}) \setminus \{f(\lambda_1), \ldots, f(\lambda_{i-1})\}$. By the assumption $\pi_0 \circ \phi = \pi_0 \circ \phi'$, the multisets $\phi^*(\mathcal{W})$ and $\phi^{\prime,*}(\mathcal{W})$ of Y coincide. It follows that we have $\phi^*(\lambda_i) = \phi^{\prime,*}(\lambda_i)$ for $i = 1, 2, 3, 4$, and as $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ form a basis of X by Lemma 4.2.3, we deduce $\phi^* = \phi^{\prime,*}$. □

Hence the conjugacy class of a homomorphism from a connected compact Lie group H to F_4 is determined by the restriction of the 26-dimensional irreducible representation to H.

4.3 Maximal proper connected subgroups

Up to conjugacy, the compact group F_4 has five maximal proper connected subgroups by [Dyn57, Theorem 5.5, Theorem 14.1]. We will recall these five subgroups in this subsection and show that there are no other maximal proper connected subgroups.

We first introduce the following notations, which will be used a lot of times in this section:

Notation 4.3.1. In this article, we use the following notations of compact Lie groups:

- For $n \geq 2$, denote by $\mathrm{SU}(n)$ the compact special unitary group with respect to the standard Hermitian form on $\mathbb{C}^n$.
- For $n \geq 3$, denote by $\mathrm{SO}(n)$ the compact special orthogonal group with respect to the standard quadratic form on $\mathbb{R}^n$, and by $\mathrm{Spin}(n)$ the compact spin group, which is a double cover of $\mathrm{SO}(n)$.
- For $n \geq 1$, denote by $\mathrm{Sp}(n)$ the *compact* symplectic group: the group of invertible $n \times n$ quaternionic matrices that preserve the standard Hermitian form

$$\langle x, y \rangle = \overline{x_1} y_1 + \cdots + \overline{x_n} y_n$$

  on $\mathbb{H}^n$, where $\mathbb{H}$ is Hamilton's quaternions.
- The group G_2 is defined as $\mathrm{Aut}(\mathbb{O}_{\mathbb{R}}, \circ)$, the automorphism group of the real octonion division algebra, which is simply connected and has trivial center.

Remark 4.3.2. The complexification of the compact symplectic group $\mathrm{Sp}(n)$ is the usual complex symplectic group $\mathrm{Sp}(2n, \mathbb{C}) = \mathrm{Sp}_{2n}(\mathbb{C})$, which is defined as the group of linear transformations of $\mathbb{C}^{2n}$ preserving the standard symplectic bilinear form.

Notation 4.3.3. We denote by μ_n the group of nth roots of unity. If m groups $G_1, \ldots, G_m$ all have a unique central subgroup isomorphic to μ_n with an embedding $\iota_i : \mu_n \hookrightarrow G_i$, we denote by μ_n^Δ the diagonal subgroup

$$\{(\iota_1(g), \ldots, \iota_m(g)) \mid g \in \mu_n\} \subset G_1 \times \cdots \times G_m.$$

Note that when $n = 2$ the embedding ι_i is unique, but when $n \geq 3$ we have to give $\iota_1, \ldots, \iota_m$ for defining μ_n^Δ.

Following Dynkin's definitions of R-subalgebras and S-subalgebras in [Dyn57, §7], we give the following definition for subgroups:

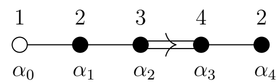
Definition 4.3.4. Let G be a connected compact Lie group and H a connected closed subgroup. We say that H is a *regular subgroup* if it is normalized by a maximal torus of G. If there is only one regular subgroup of G containing H, namely G itself, we call H an *S-subgroup*, otherwise we call it an *R-subgroup*.

Examples 4.3.5. (1) Subgroups with maximal ranks are regular.
(2) A proper regular subgroup is an R-subgroup.

(3) The principal 3-dimensional subgroups are S-subgroups by [Dyn57, Theorem 9.1].
(4) A maximal proper regular subgroup has maximal rank.

Let H be a maximal proper regular subgroup of G, i.e. if there is another regular subgroup H' of G containing H, then we have $H' = G$. The Borel-de Siebenthal theory tells us the Dynkin diagram of the root system of H is obtained by deleting an ordinary vertex with prime label from the extended Dynkin diagram of the root system of G.

For our compact group F_4, the extended Dynkin diagram is:



,

The vertex α_1 corresponds to $(\mathrm{Sp}(1) \times \mathrm{Sp}(3))\,/\mu_2^{\Delta}$, α_2 corresponds to $(\mathrm{SU}(3) \times \mathrm{SU}(3))\,/\mu_3^{\Delta}$ (we will define this μ_3^{Δ} in §4.3.3), and α_4 corresponds to $\mathrm{Spin}(9)$. The vertex α_3 corresponds to $(\mathrm{SU}(2) \times \mathrm{SU}(4))\,/\mu_2^{\Delta}$, which is also regular but not maximal since we have the embedding:

$$(\mathrm{SU}(2) \times \mathrm{SU}(4))\,/\mu_2^{\Delta} \simeq (\mathrm{Spin}(3) \times \mathrm{Spin}(6))\,/\mu_2^{\Delta} \hookrightarrow \mathrm{Spin}(9).$$

These three maximal proper regular subgroups are also maximal among proper connected subgroups of F_4, because any connected subgroup containing one of them has maximal rank and must be regular.

Besides these three regular subgroups, F_4 also admits other maximal proper connected subgroups that are not regular. A non-regular maximal connected subgroup H of F_4 must be an S-subgroup. As a subgroup of F_4 containing an S-subgroup is also an S-subgroup, it suffices to find all maximal S-subgroups of F_4.

Theorem 4.3.6. *[Dyn57, Theorem 14.1] Up to conjugacy, there are two maximal S-subgroups in F_4: the principal $\mathrm{PSU}(2)$ and $\mathrm{G}_2 \times \mathrm{SO}(3)$, where $\mathrm{PSU}(2) := \mathrm{SU}(2)/\{\pm \mathrm{id}\}$ is the adjoint group of $\mathrm{SU}(2)$.*

Putting the Borel-de Siebenthal theory and Theorem 4.3.6 together, we have:

Theorem 4.3.7. *Up to conjugacy, there are five maximal proper connected subgroups of F_4. They are respectively isomorphic to*

$$\mathrm{Spin}(9), (\mathrm{Sp}(1) \times \mathrm{Sp}(3))\,/\mu_2^{\Delta}, (\mathrm{SU}(3) \times \mathrm{SU}(3))\,/\mu_3^{\Delta}, \mathrm{G}_2 \times \mathrm{SO}(3), \textit{(principal)}\ \mathrm{PSU}(2).$$

In the rest of this subsection, we will give the explicit embeddings of these five maximal proper connected subgroups into F_4 and compute their centralizers in F_4.

4.3.1 Spin(9)

There is an involution $\sigma \in \mathrm{F}_4$ on $\mathrm{J}_{\mathbb{R}}$ defined by:

$$\sigma\ [a, b, c\,; x, y, z] = [a, b, c\,; x, -y, -z]\,, \text{ for all } a, b, c \in \mathbb{R}, x, y, z \in \mathbb{O}_{\mathbb{R}}.$$

By [Yok09, Theorem 2.9.1], the centralizer $\mathrm{C}_{\mathrm{F}_4}(\sigma)$ of σ in F_4 is also the stabilizer of $\mathrm{E}_1 = \mathrm{diag}(1, 0, 0) \in \mathrm{J}_{\mathbb{R}}$.

Lemma 4.3.8. *The group* $\mathrm{C}_{\mathrm{F}_4}(\sigma)$ *preserves respectively the subspaces*

$$\mathrm{J}_1 := \{[0, b, -b\,; x, 0, 0] \,|\, b \in \mathbb{R}, x \in \mathbb{O}_{\mathbb{R}}\}$$

and

$$\mathrm{J}_2 := \{[0, 0, 0\,; 0, y, z] \,|\, y, z \in \mathbb{O}_{\mathbb{R}}\}$$

of $\mathrm{J}_{\mathbb{R}}$.

Proof. The first subspace J_1 is exactly $\{X \in \mathrm{J}_{\mathbb{R}} \,|\, E_1 \circ X = 0, \mathrm{Tr}(X) = 0\}$ and the second subspace is $\{X \in \mathrm{J}_{\mathbb{R}} \,|\, 2E_1 \circ X = X\}$. The lemma follows from the fact that $\mathrm{C}_{\mathrm{F}_4}(\sigma)$ is the stabilizer of E_1 in F_4. □

This lemma gives the following group homomorphism:

$$\mathrm{C}_{\mathrm{F}_4}(\sigma) \to \mathrm{SO}(\mathrm{J}_1) \simeq \mathrm{SO}(9), g \mapsto g|_{\mathrm{J}_1},$$

which induce an isomorphism $\mathrm{C}_{\mathrm{F}_4}(\sigma) \simeq \mathrm{Spin}(9)$ by [Ada96, Theorem 16.7(ii)]. Since the Borel-de Siebenthal theory shows that the regular connected subgroup of type B_4 is unique up to F_4-conjugacy, so we shall thus refer to this group $\mathrm{C}_{\mathrm{F}_4}(\sigma)$ as Spin(9) in the sequel, by a slight abuse of language.

The restriction of the 26-dimensional irreducible representation (π_0, J_0) to Spin(9) is isomorphic to

$$\mathbf{1} \oplus \mathrm{V}_9 \oplus \mathrm{V}_{\mathrm{Spin}}, \tag{4.1}$$

where $\mathbf{1}$ is the trivial representation, V_9 is the standard 9-dimensional representation and $\mathrm{V}_{\mathrm{Spin}}$ is the 16-dimensional spinor module. These two representations V_9 and $\mathrm{V}_{\mathrm{Spin}}$ can be realized on J_1 and J_2 respectively.

Notation 4.3.9. To make the restriction of J_0 not too messy when it involves both direct sums and tensor products, we will replace $\oplus$ by $+$ when writing down the decomposition. For example, we write $\mathrm{J}_0|_{\mathrm{Spin}(9)}$ as $\mathbf{1} + \mathrm{V}_9 + \mathrm{V}_{\mathrm{Spin}}$.

The restriction of the adjoint representation $\mathfrak{f}_4$ of F_4 to Spin(9) is isomorphic to:

$$\wedge^2\mathrm{V}_9 + \mathrm{V}_{\mathrm{Spin}}, \tag{4.2}$$

where $\wedge^2\mathrm{V}_9$ is the adjoint representation of Spin(9).

Now we compute the centralizer of Spin(9). If an element g centralizes Spin(9), then it must commute with $\sigma \in \mathrm{Spin}(9)$. Hence $\mathrm{C}_{\mathrm{F}_4}(\mathrm{Spin}(9))$ is contained in $\mathrm{C}_{\mathrm{F}_4}(\sigma) = \mathrm{Spin}(9)$, thus it is isomorphic to the center of Spin(9), which is isomorphic to $\mathbb{Z}/2\mathbb{Z}$ and generated by σ.

Remark 4.3.10. By symmetry, the stabilizer of $\mathrm{E}_2 = \mathrm{diag}(0, 1, 0)$ (resp. $\mathrm{E}_3 = \mathrm{diag}(0, 0, 1)$) is also the centralizer of the map $[a, b, c\,; x, y, z] \mapsto [a, b, c\,; -x, y, -z]$ (resp. $[a, b, c\,; -x, -y, z]$) in F_4, and is isomorphic to Spin(9).

4.3.2 $(\mathrm{Sp}(1)\times\mathrm{Sp}(3))/\mu_2^{\Delta}$

The subalgebra of $\mathbb{O}_{\mathbb{R}}$ generated by $1, \mathrm{e}_1, \mathrm{e}_2, \mathrm{e}_4$ is isomorphic to the quaternion division algebra $\mathbb{H}$, and as a real vector space $\mathbb{O}_{\mathbb{R}}$ can be decomposed as $\mathbb{H}\oplus\mathbb{H}\mathrm{e}_5$. Using this decomposition, the conjugation on $\mathbb{O}_{\mathbb{R}}$ becomes

$$x+y\mathrm{e}_5 \mapsto \overline{x}-y\mathrm{e}_5, \text{ for all } x,y\in\mathbb{H}.$$

As $\mathrm{J}_{\mathbb{R}}=\mathrm{Herm}_3(\mathbb{O}_{\mathbb{R}})$ is the space of "Hermitian" matrices in $\mathrm{M}_3(\mathbb{O}_{\mathbb{R}})$, we embed the space $\mathrm{Herm}_3(\mathbb{H})$ of "Hermitian" matrices in $\mathrm{M}_3(\mathbb{H})$ into $\mathrm{J}_{\mathbb{R}}$ via our identification of $\mathbb{H}$ as a subalgebra of $\mathbb{O}_{\mathbb{R}}$. Then we have the following isomorphism of vector spaces:

$$\begin{aligned}\mathrm{Herm}_3(\mathbb{H})\oplus\mathbb{H}^3 &\to \mathrm{J}_{\mathbb{R}},\\ (M, a=(a_1,a_2,a_3)) &\mapsto M+[0,0,0\,;a_1\mathrm{e}_5,a_2\mathrm{e}_5,a_3\mathrm{e}_5].\end{aligned}$$

With this identification, we have an involution γ in F_4 defined as

$$\gamma(M,a)=(M,-a).$$

Proposition 4.3.11. *[Yok09, Theorem 2.11.2] Let $\varphi:\mathrm{Sp}(1)\times\mathrm{Sp}(3)\to\mathrm{GL}(\mathrm{J}_{\mathbb{R}})$ be the morphism defined as*

$$\varphi(p,A)\,(M,a)=\left(AMA^{-1},paA^{-1}\right),\ \text{for } M\in\mathrm{Herm}_3(\mathbb{H}), a\in\mathbb{H}^3.$$

Then the kernel of φ is the diagonal subgroup μ_2^{Δ} generated by γ, and the image of φ is $\mathrm{C}_{\mathrm{F}_4}(\gamma)$. In particular, φ induces an isomorphism:

$$(\mathrm{Sp}(1)\times\mathrm{Sp}(3))/\mu_2^{\Delta}\simeq\mathrm{C}_{\mathrm{F}_4}(\gamma).$$

From now on we refer to the regular connected subgroup $\mathrm{C}_{\mathrm{F}_4}(\gamma)$ as $(\mathrm{Sp}(1)\times\mathrm{Sp}(3))/\mu_2^{\Delta}$.

The restriction of the irreducible representation J_0 of F_4 to this subgroup is isomorphic to

$$\mathrm{St}\otimes\mathrm{V}_6+\mathbf{1}\otimes\mathrm{V}_{14}, \tag{4.3}$$

where St is the 2-dimensional standard representation of $\mathrm{Sp}(1)\simeq\mathrm{SU}(2)$, V_6 is the standard 6-dimensional representation of $\mathrm{Sp}(3)$ and V_{14} is the 14-dimensional irreducible representation of $\mathrm{Sp}(3)$ which satisfies $\wedge^2\mathrm{V}_3\simeq\mathrm{V}_{14}\oplus\mathbf{1}$. The first component $\mathrm{St}\otimes\mathrm{V}_6$ is realized on $\mathbb{H}^3$ and the second component $\mathbf{1}\otimes\mathrm{V}_{14}$ is realized on the trace-zero part of $\mathrm{Herm}_3(\mathbb{H})$.

The restriction of the adjoint representation $\mathfrak{f}_4$ of F_4 to $(\mathrm{Sp}(1)\times\mathrm{Sp}(3))/\mu_2^{\Delta}$ is isomorphic to

$$\mathrm{Sym}^2\,\mathrm{St}\otimes\mathbf{1}+\mathrm{St}\otimes\mathrm{V}'_{14}+\mathbf{1}\otimes\mathrm{Sym}^2\,\mathrm{V}_6, \tag{4.4}$$

where V'_{14} is another 14-dimensional irreducible representation of $\mathrm{Sp}(3)$.

By a similar argument in the case of $\mathrm{Spin}(9)$, the centralizer of $(\mathrm{Sp}(1)\times\mathrm{Sp}(3))/\mu_2^{\Delta}$ in F_4 is isomorphic to $\mathrm{Z}((\mathrm{Sp}(1)\times\mathrm{Sp}(3))/\mu_2^{\Delta})\simeq\mathbb{Z}/2\mathbb{Z}$. It is generated by the involution γ, which corresponds to $(-1,1)$ in $\mathrm{Z}(\mathrm{Sp}(1)\times\mathrm{Sp}(3))\simeq\mu_2\times\mu_2$.

Remark 4.3.12. It may help to notice that there are exactly two conjugacy classes of involutions in F_4, whose centralizers in F_4 are $\mathrm{Spin}(9)$ and $(\mathrm{Sp}(1)\times\mathrm{Sp}(3))/\mu_2^{\Delta}$ respectively.

4.3.3 $(\mathrm{SU}(3)\times\mathrm{SU}(3))/\mu_3^{\Delta}$

Take $\omega = \frac{-1+\sqrt{-3}}{2}$ and identify the center of SU(3) with μ_3 by identifying ω with the scalar matrix $\omega \mathrm{I}_3$. Then the diagonal subgroup $\mu_3^{\Delta} \subset \mathrm{SU}(3)\times\mathrm{SU}(3)$ is generated by (ω,ω).

By [Yok09, Theorem 2.12.2], the centralizer in F_4 of an order 3 element in F_4 is isomorphic to $(\mathrm{SU}(3)\times\mathrm{SU}(3))/\mu_3^{\Delta}$. As before, by an abuse of language we will refer to this subgroup as $(\mathrm{SU}(3)\times\mathrm{SU}(3))/\mu_3^{\Delta}$. Notice that the roots of the first copy of SU(3) are short roots of F_4, and those of the second copy are long roots of F_4.

Since SU(3) admits an outer automorphism, this unique (up to conjugacy) $2\mathrm{A}_2$-type subgroup $(\mathrm{SU}(3)\times\mathrm{SU}(3))/\mu_3^{\Delta}$ of F_4 has two embeddings into F_4 which are not conjugate. The restrictions of the irreducible representation J_0 along those embeddings are isomorphic to

$$\mathfrak{sl}_3\otimes\mathbf{1}+\mathrm{V}_3\otimes\mathrm{V}_3'+\mathrm{V}_3'\otimes\mathrm{V}_3 \tag{4.5}$$

and

$$\mathfrak{sl}_3\otimes\mathbf{1}+\mathrm{V}_3\otimes\mathrm{V}_3+\mathrm{V}_3'\otimes\mathrm{V}_3' \tag{4.6}$$

respectively. Here V_3 is the standard 3-dimensional representation of SU(3), V_3' is the dual representation of V_3, and $\mathfrak{sl}_3$ is the adjoint representation of SU(3).

The restriction of the adjoint representation $\mathfrak{f}_4$ of F_4 to $(\mathrm{SU}(3)\times\mathrm{SU}(3))/\mu_3^{\Delta}$ is isomorphic to

$$\mathfrak{sl}_3\otimes\mathbf{1}+\mathbf{1}\otimes\mathfrak{sl}_3+\mathrm{Sym}^2\,\mathrm{V}_3\otimes\mathrm{V}_3'+\mathrm{Sym}^2\,\mathrm{V}_3'\otimes\mathrm{V}_3 \tag{4.7}$$

or

$$\mathfrak{sl}_3\otimes\mathbf{1}+\mathbf{1}\otimes\mathfrak{sl}_3+\mathrm{Sym}^2\,\mathrm{V}_3\otimes\mathrm{V}_3+\mathrm{Sym}^2\,\mathrm{V}_3'\otimes\mathrm{V}_3'. \tag{4.8}$$

Again, we have an isomorphism $\mathrm{C}_{\mathrm{F}_4}((\mathrm{SU}(3)\times\mathrm{SU}(3))/\mu_3^{\Delta})\simeq\mathbb{Z}/3\mathbb{Z}$.

4.3.4 $\mathrm{G}_2\times\mathrm{SO}(3)$

We define an injective morphism $\iota:\mathrm{G}_2\times\mathrm{SO}(3)\hookrightarrow\mathrm{GL}(\mathrm{J}_{\mathbb{R}})$ by

$$\iota(g,O)[a,b,c\,;x,y,z]=O[a,b,c\,;g(x),g(y),g(z)]O^{-1},\text{ for all } a,b,c\in\mathbb{R}, x,y,z\in\mathbb{O}_{\mathbb{R}}, \tag{4.9}$$

by viewing $O\in\mathrm{SO}(3)$ as an element in $\mathrm{J}_{\mathbb{R}}=\mathrm{Herm}_3(\mathbb{O}_{\mathbb{R}})$ with entries in $\mathbb{R}$. This morphism is well-defined since real numbers $\mathbb{R}$ is the center of the octonion division algebra $\mathbb{O}_{\mathbb{R}}$. For any $g\in\mathrm{G}_2$ and $O\in\mathrm{SO}(3)$, the linear automorphism $\iota(g,O)$ preserves the cubic form det and the polarization I, thus ι induces an embedding of $\mathrm{G}_2\times\mathrm{SO}(3)$ into F_4. In the sequel we will refer to the image of ι as $\mathrm{G}_2\times\mathrm{SO}(3)$.

The restriction of the irreducible representation J_0 to $\mathrm{G}_2\times\mathrm{SO}(3)$ is isomorphic to

$$\mathrm{V}_7\otimes\mathrm{Sym}^2\,\mathrm{St}+\mathbf{1}\otimes\mathrm{Sym}^4\,\mathrm{St}, \tag{4.10}$$

where V_7 is the fundamental 7-dimensional representation of G_2 (the trace-zero part of $\mathbb{O}_{\mathbb{C}}$) and St denotes the standard 2-dimensional representation of SU(2). Here we use the exceptional isomorphism $\mathrm{SO}(3) \simeq \mathrm{PSU}(2) = \mathrm{SU}(2)/\mu_2$ to view odd dimensional irreducible representations $\mathrm{Sym}^{2n}\mathrm{St}, n \in \mathbb{N}$ of $\mathrm{SU}(2)$ as irreducible representations of $\mathrm{SO}(3)$. The first component $\mathrm{V}_7 \otimes \mathrm{Sym}^2 \mathrm{St}$ is realized on the space

$$\{[0,0,0\,;x,y,z]\,|\,x,y,z \in \mathbb{O}_{\mathbb{R}}, \mathrm{Tr}(x)=\mathrm{Tr}(y)=\mathrm{Tr}(z)=0\},$$

and the second component $\mathbf{1} \otimes \mathrm{Sym}^4 \mathrm{St}$ is realized on the space

$$\{[a,b,c\,;x,y,z]\,|\,a,b,c,x,y,z \in \mathbb{R}, a+b+c=0\}.$$

The restriction of the adjoint representation $\mathfrak{f}_4$ of F_4 to $\mathrm{G}_2 \times \mathrm{SO}(3)$ is isomorphic to

$$\mathfrak{g}_2 \otimes \mathbf{1} + \mathrm{V}_7 \otimes \mathrm{Sym}^4 \mathrm{St} + \mathbf{1} \otimes \mathrm{Sym}^2 \mathrm{St}, \tag{4.11}$$

where $\mathfrak{g}_2$ is the adjoint representation of G_2.

Proposition 4.3.13. *The centralizer of* $\mathrm{G}_2 \times \mathrm{SO}(3)$ *in* F_4 *is trivial.*

Proof. Let g be an element in $\mathrm{C}_{\mathrm{F}_4}(\mathrm{G}_2 \times \mathrm{SO}(3))$. Because the image of $\mathrm{diag}(1,-1,-1) \in \mathrm{SO}(3)$ in F_4 is the involution σ defined in §4.3.1, g lies in $\mathrm{C}_{\mathrm{F}_4}(\sigma)$, thus it stabilizes E_1. By Remark 4.3.10, we also have g stabilizes E_2 and E_3 respectively. According to [Ada96, Theorem 16.7(iii), Lemma 15.15], g is an element of the form

$$[a,b,c\,;x,y,z] \mapsto [a,b,c\,;\alpha(x),\beta(y),\gamma(z)], \text{ for all } a,b,c \in \mathbb{R}, x,y,z \in \mathbb{O}_{\mathbb{R}},$$

where $\alpha,\beta,\gamma \in \mathrm{SO}(\mathbb{O}_{\mathbb{R}})$ satisfy

$$\overline{\alpha(x)\beta(y)} = \gamma(\overline{xy}) \text{ for all } x,y \in \mathbb{O}_{\mathbb{R}}. \tag{4.12}$$

The image of $\left(\begin{smallmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{smallmatrix}\right) \in \mathrm{SO}(3)$ in F_4 is the map

$$[a,b,c\,;x,y,z] \mapsto [a,c,b\,;-\overline{x},-\overline{z},\overline{y}].$$

The fact that it commutes with g implies that $\alpha(\overline{x}) = \overline{\alpha(x)}$ and $\beta(\overline{x}) = \overline{\gamma(x)}$ for all $x \in \mathbb{O}_{\mathbb{R}}$. By symmetry we get $\alpha=\beta=\gamma$ and (4.12) shows that

$$\alpha(x)\alpha(y) = \overline{\alpha(\overline{xy})} = \alpha(\overline{\overline{xy}}) = \alpha(xy), \text{ for all } x,y \in \mathbb{O}_{\mathbb{R}}.$$

Hence $\alpha \in \mathrm{G}_2$ and we have proved that $\mathrm{C}_{\mathrm{F}_4}(\mathrm{SO}(3)) = \mathrm{G}_2$, thus the centralizer of $\mathrm{G}_2 \times \mathrm{SO}(3)$ in F_4 is the center of G_2, which is trivial. ☐

4.3.5 The principal PSU(2)

The image of the *principal embedding* from $\mathrm{SU}(2)$ into F_4, in the sense of [CM92, Theorem 4.1.6], is also a maximal proper connected subgroup of F_4. The restriction of the irreducible representation J_0 to this $\mathrm{SU}(2)$ is isomorphic to

$$\mathrm{Sym}^8 \mathrm{St} + \mathrm{Sym}^{16} \mathrm{St},$$

where St is the standard 2-dimensional representation of $\mathrm{SU}(2)$. This implies that the image is isomorphic to $\mathrm{PSU}(2)$, and we call it the principal $\mathrm{PSU}(2)$ of F_4.

By the general property of principal embeddings, its centralizer is the center of F_4. It is well-known that the center of F_4 is trivial.

4.4 Classification of A_1-subgroups

In this subsection we will classify A_1-*subgroups* of F_4, i.e. subgroups that are isomorphic to $\mathrm{SU}(2)$ or $\mathrm{PSU}(2)$. By [Dyn57, Theorem 9.3] every A_1-subgroup X of F_4 is either the principal $\mathrm{PSU}(2)$ or an R-subgroup, i.e. X is contained in some proper regular subgroup of F_4. When X is an R-subgroup, up to conjugacy it is contained in one of the three regular maximal proper connected subgroups of F_4 we have found in §4.3. All these three regular subgroups arise from classical groups, thus their A_1-subgroups are well-known.

By Proposition 4.2.1, a conjugacy class of A_1-groups of F_4 is determined uniquely by the restriction of the 26-dimensional representation J_0 to it.

Notation 4.4.1. An isomorphism class of n-dimensional representation of $\mathrm{SU}(2)$ gives a partition of the integer n. We will use the notation $[N^{k_N}, (N-1)^{k_{N-1}}, \ldots, 2^{k_2}, 1^{k_1}]$, where $k_N \neq 0$ and $\sum_{i=1}^{N} ik_i = n$, for a partition of n. For example, the restriction of J_0 to the principal $\mathrm{PSU}(2)$ is isomorphic to $\mathrm{Sym}^8 \mathrm{St} + \mathrm{Sym}^{16} \mathrm{St}$, thus we index this A_1-subgroup by the partition $[17, 9]$ of $\dim \mathrm{J}_0 = 26$.

4.4.1 A_1-subgroups of $\mathrm{Spin}(9)$

We start from A_1-subgroups of $\mathrm{SO}(9)$. According to [CM92, Theorem 5.1.2], the conjugacy classes of morphisms $\mathrm{SU}(2) \to \mathrm{SO}(9)$ are in bijection with partitions of 9 in which each even number appears even times.

Lemma 4.4.2. *(1) There are* 12 *different conjugacy classes of* A_1*-subgroups of* $\mathrm{Spin}(9)$*, which correspond to the following partitions of* 9*:*

$$[9], [7, 1^2], [5, 3, 1], [5, 2^2], [5, 1^4], [4^2, 1], [3^3], [3^2, 1^3], [3, 2^2, 1^2], [3, 1^6], [2^4, 1], [2^2, 1^5].$$

(2) There are 10 *different conjugacy classes of* A_1*-subgroups of* F_4 *that are contained in the subgroup* $\mathrm{Spin}(9)$ *given in §4.3.1. The restrictions of the* 26*-dimensional irreducible representation* J_0 *of* F_4 *to these* A_1*-subgroups correspond to the following partitions of* 26*:*

$$\begin{gathered} [11, 9, 5, 1], [7^3, 1^5], [5^3, 3^3, 1^2], [3^6, 1^8], \\ [5^2, 4^2, 3, 2^2, 1], [5, 4^4, 1^5], [4^2, 3^3, 2^4, 1], [3^3, 2^6, 1^5], [3, 2^8, 1^7], [2^6, 1^{14}]. \end{gathered} \tag{4.13}$$

Proof. By the lifting property of covering maps and the fact that $\mathrm{SU}(2)$ is simply connected, every A_1-subgroup of $\mathrm{SO}(9)$ is lifted uniquely to an A_1-subgroup of $\mathrm{Spin}(9)$. The assertion (1) follows directly from [CM92, Theorem 5.1.2], and the assertion (2) follows from the equivalence (4.1). □

The A_1-subgroups in the first row of (4.13) are isomorphic to $\mathrm{PSU}(2)$ and the A_1-subgroups in the second row are isomorphic to $\mathrm{SU}(2)$.

4.4.2 A_1-subgroups of $(\mathrm{Sp}(1) \times \mathrm{Sp}(3))/\mu_2^{\Delta}$

We apply the same argument for A_1-subgroups of $(\mathrm{Sp}(1) \times \mathrm{Sp}(3))/\mu_2^{\Delta}$. By [CM92, Theorem 5.1.3], the set of conjugacy classes of morphisms $\mathrm{SU}(2) \to \mathrm{Sp}(3)$ are in bijection with partitions of 6 in which each odd number appears even times.

Lemma 4.4.3. *(1) There are* 7 *different conjugacy classes of* A_1*-subgroups of* $\mathrm{Sp}(3)$*, which correspond to the following partitions of* 6*:*

$$[6], [4,2], [4,1^2], [3^2], [2^3], [2^2,1^2], [2,1^4].$$

(2) There are 11 *different conjugacy classes of* A_1*-subgroups of* F_4 *that are contained in the subgroup* $(\mathrm{Sp}(1)\times\mathrm{Sp}(3))/\mu_2^\Delta$ *given in §4.4.2. The restrictions of the* 26*-dimensional irreducible representation* J_0 *of* F_4 *to these* A_1*-subgroups correspond to the following partitions of* 26*:*

$$\begin{gathered}[9,7,5^2],[5^3,3^3,1^2],[5,3^7],[3^6,1^8],\\ [9,6^2,5],[5^2,4^2,3,2^2,1],[5,4^4,1^5],[5,4^2,3^3,2^2],[3^3,2^6,1^5],[3,2^8,1^7],[2^6,1^{14}].\end{gathered}\tag{4.14}$$

Proof. The assertion (1) follows directly from [CM92, Theorem 5.1.3]. A morphism from $\mathrm{SU}(2)$ to $(\mathrm{Sp}(1)\times\mathrm{Sp}(3))/\mu_2^\Delta$ arises from the product of two morphisms $\mathrm{SU}(2)\to\mathrm{Sp}(1)$ and $\mathrm{SU}(2)\to\mathrm{Sp}(3)$. The assertion (2) follows from the equivalence (4.3). □

The A_1-subgroups in the first row of (4.14) are isomorphic to $\mathrm{PSU}(2)$ and the A_1-subgroups in the second row are isomorphic to $\mathrm{SU}(2)$.

4.4.3 A_1 subgroups of $(\mathrm{SU}(3)\times\mathrm{SU}(3))/\mu_3^\Delta$

The restriction of the standard representation V_3 of $\mathrm{SU}(3)$ to an A_1-subgroup of $\mathrm{SU}(3)$ can only be $[3]$ or $[2,1]$. By the equivalences (4.5) and (4.8), we have the following result:

Lemma 4.4.4. *There are* 8 *different conjugacy classes of* A_1*-subgroups of* F_4 *that are contained in the subgroup* $(\mathrm{SU}(3)\times\mathrm{SU}(3))/\mu_3^\Delta$ *given in §4.3.3. The restrictions of the* 26*-dimensional irreducible representation* J_0 *of* F_4 *to these* A_1*-subgroups correspond to the following partitions of* 26*:*

$$\begin{gathered}[5^3,3^3,1^2],[5,3^7],[3^6,1^8]\\ [5,4^2,3^3,2^2],[4^2,3^3,2^4,1],[3^3,2^6,1^5],[3,2^8,1^7],[2^6,1^{14}].\end{gathered}\tag{4.15}$$

The A_1-subgroups in the first row of (4.15) are isomorphic to $\mathrm{PSU}(2)$ and subgroups in the second row are isomorphic to $\mathrm{SU}(2)$.

4.4.4 Conclusion

Now we have enumerated (up to conjugacy) all A_1-subgroups of F_4 and indexed them by the restriction of the 26-dimensional irreducible representation J_0 of F_4.

Proposition 4.4.5. *(1) There are* 7 *conjugacy classes of subgroups of* F_4 *that are isomorphic to* $\mathrm{PSU}(2)$*, corresponding to the following partitions of* 26*:*

$$[17,9],[11,9,5,1],[9,7,5^2],[7^3,1^5],[5^3,3^3,1^2],[5,3^7],[3^6,1^8].$$

(2) There are 7 *conjugacy classes of subgroups of* F_4 *that are isomorphic to* $\mathrm{SU}(2)$*, corresponding to the following partitions of* 26*:*

$$[9,6^2,5],[5^2,4^2,3,2^2,1],[5,4^4,1^5],[5,4^2,3^3,2^2],[4^2,3^3,2^4,1],[3^3,2^6,1^5],[3,2^8,1^7],[2^6,1^{14}].$$

The theory of Jacobson-Morozov shows that the set of conjugacy classes of morphisms $\mathrm{SU}(2) \to \mathrm{F}_4$ is in bijection with the set of nilpotent orbits of the semisimple Lie algebra $\mathfrak{f}_4$. The nilpotent orbits of $\mathfrak{f}_4$ are labeled in [CM92, §8.4], and we will use the same labelings for A_1-subgroups of F_4:

Label	Restriction of J_0	Label	Restriction of J_0	Label	Restriction of J_0
A_1	$[2^6, 1^{14}]$	$\mathrm{A}_2 + \widetilde{\mathrm{A}_1}$	$[4^2, 3^3, 2^4, 1]$	B_3	$[7^3, 1^5]$
$\widetilde{\mathrm{A}_1}$	$[3, 2^8, 1^7]$	B_2	$[5, 4^4, 1^5]$	C_3	$[9, 6^2, 5]$
$\mathrm{A}_1 + \widetilde{\mathrm{A}_1}$	$[3^3, 2^6, 1^5]$	$\widetilde{\mathrm{A}_2} + \mathrm{A}_1$	$[5, 4^2, 3^3, 2^2]$	$\mathrm{F}_4(a_2)$	$[9, 7, 5^2]$
A_2	$[3^6, 1^8]$	$\mathrm{C}_3(a_1)$	$[5^2, 4^2, 3, 2^2, 1]$	$\mathrm{F}_4(a_1)$	$[11, 9, 5, 1]$
$\widetilde{\mathrm{A}_2}$	$[5, 3^7]$	$\mathrm{F}_4(a_3)$	$[5^3, 3^3, 1^2]$	F_4	$[17, 9]$

Table 2: Labels of A_1-subgroups of F_4

Notation 4.4.6. With Table 2, for a conjugacy class of A_1-subgroups of F_4, we have two ways to refer to it. For example, for the conjugacy class of principal $\mathrm{PSU}(2)$, we call it the class $[17, 9]$ or the class with label F_4.

4.4.5 Centralizers

The next thing we are going to do is to compute the centralizer, or the neutral component of the centralizer, of each A_1-subgroup of F_4. In the following paragraphs, we choose a representative $\mathrm{SU}(2) \to \mathrm{F}_4$ for each conjugacy class of A_1-subgroups, whose image is denoted by X, and then determine $\mathrm{C}_{\mathrm{F}_4}(X)$ or $\mathrm{C}_{\mathrm{F}_4}(X)^\circ$.

The following lemma will be used when computing the centralizer of a subgroup in F_4:

Lemma 4.4.7. *Let G be the quotient of a Lie group G_0 by a finite central subgroup Γ. If H_0 is a connected subgroup of G_0, whose image in G is denoted by H, then the inverse image of $\mathrm{C}_G(H)$ in G_0 is $\mathrm{C}_{G_0}(H_0)$ and $\mathrm{C}_G(H) \simeq \mathrm{C}_{G_0}(H_0)/\Gamma$.*

Proof. It suffices to prove that any $g_0 \in G_0$ whose image g lies in $\mathrm{C}_G(H)$ centralizes H_0. For any $h_0 \in H_0$ with image h in H, we have $ghg^{-1}h^{-1} = 1$ in G, thus $g_0h_0g_0^{-1}h_0^{-1} \in \Gamma$. The continuous map $\varphi : H_0 \to \Gamma, h_0 \mapsto g_0h_0g_0^{-1}h_0^{-1}$ for $h_0 \in H_0$ must be constant, because H_0 is connected and Γ is discrete as a finite group. The map φ sends $1 \in H_0$ to $1 \in \Gamma$, thus $\varphi \equiv 1$, which implies that g_0 centralizes H_0 in G_0. □

In some cases we can not compute the centralizer $\mathrm{C}_{\mathrm{F}_4}(X)$ easily, then we use the following lemma to determine its neutral component $\mathrm{C}_{\mathrm{F}_4}(X)^\circ$:

Lemma 4.4.8. *Let H be a connected subgroup of a compact Lie group G, and d the multiplicity of $\mathbf{1}$ in the restriction of the adjoint representation $\mathfrak{g}$ of G to H. If there is a d-dimensional connected subgroup C of $\mathrm{C}_G(H)$, then we have $\mathrm{C}_G(H)^\circ = C$. In particular, the centralizer $\mathrm{C}_G(H)$ is a finite group when $d = 0$.*

Proof. As subalgebras of $\mathfrak{g}$, the Lie algebra $\mathrm{Lie}(\mathrm{C}_G(H)^\circ)$ of $\mathrm{C}_G(H)^\circ$ is contained in

$$\mathrm{C}_{\mathfrak{g}}(H) := \{X \in \mathfrak{g} \,|\, \mathrm{Ad}(g)X = X \text{ for all } g \in H_{\mathbb{C}}\}\,,$$

where $H_{\mathbb{C}}$ is the complexification of H. The dimension of $\mathrm{C}_{\mathfrak{g}}(H)$ equals the multiplicity d of $\mathbf{1}$ in $\mathfrak{g}|_H$.

Let $\mathfrak{c}$ be the complexified Lie algebra of C. We have the inclusions $\mathfrak{c} \subset \mathrm{Lie}(\mathrm{C}_G(H)^\circ) \subset \mathrm{C}_{\mathfrak{g}}(H)$. Since $\dim \mathfrak{c} = d = \dim \mathrm{C}_{\mathfrak{g}}(H)$, these three subspaces of $\mathfrak{g}$ are equal. It is well known that a connected Lie group is generated by a neighborhood of the identity element, thus the connected subgroups C and $\mathrm{C}_G(H)^\circ$ of G coincide. $\square$

4.4.5.1 $[17, 9]$ We choose X to be the principal $\mathrm{PSU}(2)$ in F_4, whose centralizer in F_4 is trivial.

4.4.5.2 $[11, 9, 5, 1]$ We choose X to be the principal $\mathrm{PSU}(2)$ of the $\mathrm{Spin}(9)$ given in §4.3.1. The restriction of the adjoint representation $\mathfrak{f}_4$ of F_4 to X corresponds to the partition $[15, 11^2, 7, 5, 3]$ of 52, which implies that $\mathrm{C}_{\mathrm{F}_4}(X)$ is a finite group by Lemma 4.4.8.

4.4.5.3 $[9, 7, 5^2]$ We choose X to be the principal $\mathrm{PSU}(2)$ of the $(\mathrm{Sp}(1) \times \mathrm{Sp}(3))\,/\mu_2^\Delta$ given in §4.3.2. The restriction of the adjoint representation $\mathfrak{f}_4$ to X corresponds to the partition $[11^2, 9, 7, 5, 3^3]$ of 52, thus $\mathrm{C}_{\mathrm{F}_4}(X)$ is a finite group by Lemma 4.4.8.

4.4.5.4 $[7^3, 1^5]$ We choose X to be the principal $\mathrm{PSU}(2)$ of the factor G_2 in the subgroup $\mathrm{G}_3 \times \mathrm{SO}(3)$ given in §4.3.4. The other factor $\mathrm{SO}(3)$ of $\mathrm{G}_2 \times \mathrm{SO}(3)$ centralizes this A_1-subgroup X. The restriction of the adjoint representation $\mathfrak{f}_4$ of F_4 to X corresponds to the partition $[11, 7^5, 3, 1^3]$ of 52, thus $\mathrm{C}_{\mathrm{F}_4}(X)^\circ$ is the $\mathrm{SO}(3)$ in $\mathrm{G}_2 \times \mathrm{SO}(3)$ by Lemma 4.4.8, which is in the class $[5, 3^7]$ and labeled by $\widetilde{\mathrm{A}_2}$.

4.4.5.5 $[5^3, 3^3, 1^2]$ We choose X to be the principal $\mathrm{PSU}(2)$ of the $(\mathrm{SU}(3) \times \mathrm{SU}(3))\,/\mu_3^\Delta$ given in §4.3.3. The restriction of the adjoint representation $\mathfrak{f}_4$ of F_4 to X corresponds to the partition $[7^2, 5^4, 3^6]$ of 52, thus $\mathrm{C}_{\mathrm{F}_4}(X)$ is a finite group by Lemma 4.4.8. The center of $(\mathrm{SU}(3) \times \mathrm{SU}(3))\,/\mu_3^\Delta$, which is a cyclic group of order 3, is contained in $\mathrm{C}_{\mathrm{F}_4}(X)$.

4.4.5.6 $[5, 3^7]$ We choose X to be the factor $\mathrm{SO}(3)$ in the subgroup $\mathrm{G}_2 \times \mathrm{SO}(3)$ of F_4 given in §4.3.4. In the proof of Proposition 4.3.13, we have shown that the centralizer $\mathrm{C}_{\mathrm{F}_4}(X)$ is the other factor G_2.

4.4.5.7 $[3^6, 1^8]$ We choose X to be the principal $\mathrm{PSU}(2)$ of the second copy of $\mathrm{SU}(3)$ in the subgroup $(\mathrm{SU}(3) \times \mathrm{SU}(3))\,/\mu_3^\Delta$ given in §4.3.3. The first copy of $\mathrm{SU}(3)$ centralizes X and has dimension 8. The restriction of the adjoint representation $\mathfrak{f}_4$ of F_4 to X corresponds to the partition $[5, 3^{13}, 1^8]$ of 52, thus $\mathrm{C}_{\mathrm{F}_4}(X)^\circ$ is the first copy of $\mathrm{SU}(3)$ in $(\mathrm{SU}(3) \times \mathrm{SU}(3))\,/\mu_3^\Delta$ by Lemma 4.4.8, whose roots are short roots of F_4.

4.4.5.8 $[9, 6^2, 5]$ We choose X_0 to be the principal $\mathrm{SU}(2)$ of $\mathrm{Sp}(3)$, and X to be the image of X_0 in the subgroup $(\mathrm{Sp}(1) \times \mathrm{Sp}(3))/\mu_2^\Delta$ given in §4.3.2. The group $(\mathrm{Sp}(1) \times \mathrm{Sp}(3))/\mu_2^\Delta$ is defined as $\mathrm{C}_{\mathrm{F}_4}(\gamma)$, where γ is an involution in F_4 and is the image of $(1, -\mathrm{I}_3) \in \mathrm{Sp}(1) \times \mathrm{Sp}(3)$ in the quotient group.

Since X contains the element γ, the centralizer of X in F_4 is contained in $\mathrm{C}_{\mathrm{F}_4}(\gamma) = (\mathrm{Sp}(1) \times \mathrm{Sp}(3))/\mu_2^\Delta$, thus $\mathrm{C}_{\mathrm{F}_4}(X) = \mathrm{C}_{(\mathrm{Sp}(1)\times\mathrm{Sp}(3))/\mu_2^\Delta}(X)$. By Lemma 4.4.7, we have:

$$\mathrm{C}_{(\mathrm{Sp}(1)\times\mathrm{Sp}(3))/\mu_2^\Delta}(X) = \mathrm{C}_{\mathrm{Sp}(1)\times\mathrm{Sp}(3)}(1 \times X_0)/\mu_2^\Delta = (\mathrm{Sp}(1) \times \mathrm{Z}(\mathrm{Sp}(3)))/\mu_2^\Delta \simeq \mathrm{Sp}(1).$$

Hence $\mathrm{C}_{\mathrm{F}_4}(X)$ is an A_1-subgroup in the class $[2^6, 1^{14}]$ and labeled by A_1.

4.4.5.9 $[5^2, 4^2, 3, 2^2, 1]$ We choose X_0 to be the image of

$$\mathrm{SU}(2) \hookrightarrow \mathrm{Sp}(1) \times \mathrm{Sp}(2) \hookrightarrow \mathrm{Sp}(3),$$

where the first arrow is the principal morphism of $\mathrm{Sp}(1) \times \mathrm{Sp}(2)$, and the second is defined as $(x, A) \mapsto \left(\begin{smallmatrix} x & 0 \\ 0 & A \end{smallmatrix}\right)$, for any $x \in \mathrm{Sp}(1), A \in \mathrm{Sp}(2)$. Let X be the image of X_0 in $(\mathrm{Sp}(1) \times \mathrm{Sp}(3))/\mu_2^\Delta = \mathrm{C}_{\mathrm{F}_4}(\gamma)$.

The element γ corresponds to $(1, -\mathrm{I}_3)$ in $\mathrm{Sp}(1) \times \mathrm{Sp}(3)$, thus it is contained in X, so $\mathrm{C}_{\mathrm{F}_4}(X) \subset \mathrm{C}_{\mathrm{F}_4}(\gamma)$ and $\mathrm{C}_{\mathrm{F}_4}(X) = \mathrm{C}_{(\mathrm{Sp}(1)\times\mathrm{Sp}(3))/\mu_2^\Delta}(X)$. Again by Lemma 4.4.7, we have:

$$\mathrm{C}_{(\mathrm{Sp}(1)\times\mathrm{Sp}(3))/\mu_2^\Delta}(X) = \mathrm{C}_{\mathrm{Sp}(1)\times\mathrm{Sp}(3)}(1 \times X_0)/\mu_2^\Delta = (\mathrm{Sp}(1) \times \langle\gamma_1\rangle \times \langle\gamma_2\rangle)/\mu_2^\Delta,$$

where $\gamma_1 = \left(\begin{smallmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right)$ and $\gamma_2 = \left(\begin{smallmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{smallmatrix}\right)$ are two order 2 elements in $\mathrm{Sp}(3)$. Hence $\mathrm{C}_{\mathrm{F}_4}(X)$ is the product of $\mathrm{Sp}(1)$ and an order 2 group, and this A_1-subgroup $\mathrm{Sp}(1)$ is in the class $[2^6, 1^{14}]$ and labeled by A_1.

4.4.5.10 $[5, 4^4, 1^5]$ We choose a morphism:

$$\mathrm{SU}(2) \hookrightarrow \mathrm{Spin}(5) \hookrightarrow \mathrm{Spin}(5) \times \mathrm{Spin}(4) \to \mathrm{Spin}(9) \hookrightarrow \mathrm{F}_4,$$

where the first arrow is the principal morphism of $\mathrm{Spin}(5)$, and the subgroup $\mathrm{Spin}(9)$ of F_4 is defined as $\mathrm{C}_{\mathrm{F}_4}(\sigma)$ in §4.3.1. This morphism is injective since the factor $\mathrm{Spin}(5)$ has zero intersection with the kernel of $\mathrm{Spin}(5) \times \mathrm{Spin}(4) \to \mathrm{Spin}(9)$, and we denote its image by X.

The element σ defined in §4.3.1 is contained in X, hence the centralizer of X in F_4 is contained in $\mathrm{Spin}(9)$, thus $\mathrm{C}_{\mathrm{F}_4}(X) = \mathrm{C}_{\mathrm{Spin}(9)}(X)$. Denote the natural projection $\mathrm{Spin}(9) \to \mathrm{SO}(9)$ by p. The centralizer of $p(X)$ in $\mathrm{SO}(9)$ is $\mathrm{SO}(4)$, the image of $\mathrm{Spin}(4)$ under p. By Lemma 4.4.7, we have

$$\mathrm{C}_{\mathrm{Spin}(9)}(X) = p^{-1}(\mathrm{SO}(4)) = \mathrm{Spin}(4) \simeq \mathrm{SU}(2) \times \mathrm{SU}(2),$$

and as a result $\mathrm{C}_{\mathrm{F}_4}(X)$ is the product of two A_1-subgroups in the class $[2^6, 1^{14}]$.

4.4.5.11 $[5, 4^2, 3^3, 2^2]$ We choose an embedding:

$$\mathrm{SU}(2) \hookrightarrow \mathrm{Sp}(1) \times \mathrm{SO}(3) \hookrightarrow \mathrm{Sp}(1) \times \mathrm{Sp}(3),$$

where the first arrow is the principal morphism of $\mathrm{Sp}(1)\times\mathrm{SO}(3)$, and the embedding $\mathrm{SO}(3) \to \mathrm{Sp}(3)$ is given by viewing an orthogonal 3×3 matrix as an matrix in $\mathrm{GL}(3, \mathbb{H})$ preserving the standard Hermitian form on $\mathbb{H}^3$. Let X_0 be the image of this embedding, and X the image of X in the subgroup $(\mathrm{Sp}(1) \times \mathrm{Sp}(3))/\mu_2^\Delta = \mathrm{C}_{\mathrm{F}_4}(\gamma)$ of F_4 given in §4.3.2.

The group X_0 contains $(-1, \mathrm{I}_3)$, thus the element γ is contained in X. So the centralizer $\mathrm{C}_{\mathrm{F}_4}(X)$ is contained in $\mathrm{C}_{\mathrm{F}_4}(\gamma)$ and $\mathrm{C}_{\mathrm{F}_4}(X) = \mathrm{C}_{(\mathrm{Sp}(1)\times\mathrm{Sp}(3))/\mu_2^\Delta}(X)$. By Lemma 4.4.7, we have

$$\mathrm{C}_{(\mathrm{Sp}(1)\times\mathrm{Sp}(3))/\mu_2^\Delta}(X) = \left(\mathrm{Z}(\mathrm{Sp}(1)) \times \mathrm{C}_{\mathrm{Sp}(3)}(\mathrm{SO}(3))\right)/\mu_2^\Delta \simeq \mathrm{C}_{\mathrm{Sp}(3)}\left(\mathrm{SO}(3)\right).$$

A 3×3 matrix in $\mathrm{Sp}(3)$ commutes with all elements in $\mathrm{SO}(3)$ if and only if it is a scalar matrix, thus it must be of the form $h \cdot \mathrm{I}_3$ for some norm 1 element $h \in \mathbb{H}$. Hence $C_{\mathrm{F}_4}(X) \simeq \mathrm{Sp}(1)$ is an A_1-subgroup in the class $[3^3, 2^6, 1^5]$ and labeled by $\mathrm{A}_1 + \widetilde{\mathrm{A}_1}$.

4.4.5.12 $[4^2, 3^3, 2^4, 1]$ We choose a morphism:

$$\mathrm{Spin}(3) \hookrightarrow \mathrm{Spin}(3) \times \mathrm{Spin}(3) \times \mathrm{Spin}(3) \to \mathrm{Spin}(9) = \mathrm{C}_{\mathrm{F}_4}(\sigma) \hookrightarrow \mathrm{F}_4,$$

where the first arrow is the diagonal embedding. This is also an embedding and we denote its image in F_4 by X.

Again we have $\mathrm{C}_{\mathrm{F}_4}(X) = \mathrm{C}_{\mathrm{Spin}(9)}(X)$, and by Lemma 4.4.7, the centralizer of X in $\mathrm{Spin}(9)$ is the inverse image in $\mathrm{Spin}(9)$ of the subgroup

$$\left\{ \begin{pmatrix} a_{11}\mathrm{I}_3 & a_{12}\mathrm{I}_3 & a_{13}\mathrm{I}_3 \\ a_{21}\mathrm{I}_3 & a_{22}\mathrm{I}_3 & a_{23}\mathrm{I}_3 \\ a_{31}\mathrm{I}_3 & a_{32}\mathrm{I}_3 & a_{33}\mathrm{I}_3 \end{pmatrix} \middle| \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \in \mathrm{SO}(3) \right\}$$

of $\mathrm{SO}(9)$. Hence $\mathrm{C}_{\mathrm{F}_4}(X) \simeq \mathrm{Spin}(3)$ is also an A_1-subgroup in the class $[4^2, 3^3, 2^4, 1]$.

4.4.5.13 $[3^3, 2^6, 1^5]$ We denote by X_0 the image of $\mathrm{Sp}(1) \hookrightarrow \mathrm{Sp}(3)$ given by $h \mapsto h\mathrm{I}_3$, and by X the image of X under the embedding of $\mathrm{Sp}(3)$ into the group $(\mathrm{Sp}(1) \times \mathrm{Sp}(3))/\mu_2^\Delta = \mathrm{C}_{\mathrm{F}_4}(\gamma)$ given in §4.3.2.

The element $\gamma = (1, -\mathrm{I}_3)$ (modulo μ_2^Δ) is contained in X, so the centralizer $\mathrm{C}_{\mathrm{F}_4}(X)$ equals $\mathrm{C}_{(\mathrm{Sp}(1)\times\mathrm{Sp}(3))/\mu_2^\Delta}(X)$. By Lemma 4.4.7, we have

$$\mathrm{C}_{(\mathrm{Sp}(1)\times\mathrm{Sp}(3))/\mu_2^\Delta}(X) = \mathrm{C}_{\mathrm{Sp}(1)\times\mathrm{Sp}(3)}(1 \times X_0)/\mu_2^\Delta = \left(\mathrm{Sp}(1) \times \mathrm{C}_{\mathrm{Sp}(3)}(X_0)\right)/\mu_2^\Delta.$$

A 3×3 matrix $A \in \mathrm{Sp}(3)$ commutes with $h\mathrm{I}_3$ for all norm 1 quaternions h, if and only if all entries of A are real. Hence $\mathrm{C}_{\mathrm{Sp}(3)}(X_0) = \mathrm{GL}(3, \mathbb{R}) \cap \mathrm{Sp}(3) = \mathrm{O}(3)$, and as a result $\mathrm{C}_{\mathrm{F}_4}(X) \simeq \mathrm{Sp}(1) \times \mathrm{SO}(3)$ is the product of two A_1-subgroups in the classes $[2^6, 1^{14}]$ and $[5, 3^7]$ respectively. These two A_1-subgroups are labeled by A_1 and $\widetilde{\mathrm{A}_2}$ respectively.

4.4.5.14 $[3, 2^8, 1^7]$ We choose a morphism:

$$\mathrm{Spin}(3) \hookrightarrow \mathrm{Spin}(3) \times \mathrm{Spin}(6) \to \mathrm{Spin}(9) = \mathrm{C}_{\mathrm{F}_4}(\sigma) \hookrightarrow \mathrm{F}_4,$$

which is injective, and denote by X its image in F_4.

The element σ is contained in X, thus $\mathrm{C}_{\mathrm{F}_4}(X) = \mathrm{C}_{\mathrm{Spin}(9)}(X)$. Again by Lemma 4.4.7, this centralizer is the group $\mathrm{Spin}(6)$ in the morphism we choose.

4.4.5.15 $[2^6, 1^{14}]$ We choose X to be the factor $\mathrm{Sp}(1)$ in the $(\mathrm{Sp}(1) \times \mathrm{Sp}(3))/\mu_2^{\Delta}$ given in §4.3.2. Using Lemma 4.4.7, we obtain that the centralizer $\mathrm{C}_{\mathrm{F}_4}(X)$ is the other factor $\mathrm{Sp}(3)$.

4.5 Connected simple subgroups

In this subsection, we will classify connected simple subgroups of F_4 whose ranks are larger than 1, and then determine their centralizers in F_4.

Let H be a proper connected simple subgroup of F_4 whose rank is larger than 1. It is (up to conjugacy) contained in one of the following four maximal proper connected subgroups classified in §4.3:

$$\mathrm{Spin}(9), (\mathrm{Sp}(1) \times \mathrm{Sp}(3))/\mu_2^{\Delta}, (\mathrm{SU}(3) \times \mathrm{SU}(3))/\mu_3^{\Delta}, \mathrm{G}_2 \times \mathrm{SO}(3).$$

Moreover, by [Dyn57, Theorem 14.2] the group F_4 has no simple S-subgroup except the principal $\mathrm{PSU}(2)$, so we have:

Lemma 4.5.1. *Let H be a proper connected simple subgroup of F_4 with $\mathrm{rank}\, H \geq 2$, then up to conjugacy H is contained in one of the following fixed subgroups of F_4:*

$$\mathrm{Spin}(9), \mathrm{Sp}(3), (\mathrm{SU}(3) \times \mathrm{SU}(3))/\mu_3^{\Delta}.$$

The possible Lie types for H are:

$$\mathrm{A}_2, \mathrm{A}_3, \mathrm{A}_4, \mathrm{B}_2, \mathrm{B}_3, \mathrm{B}_4, \mathrm{C}_3, \mathrm{C}_4, \mathrm{D}_4, \mathrm{G}_2.$$

Proposition 4.5.2. *There are no connected subgroups of F_4 whose Lie type is A_4 or C_4.*

Proof. Suppose that F_4 admits a connected subgroup H with type A_4 or C_4. Since $\mathrm{rank}(H) = 4$, by Lemma 4.5.1 there exists an embedding of H into $\mathrm{Spin}(9)$.

The case that H is of type C_4 is impossible, because $\dim H = 36 = \dim \mathrm{Spin}(9)$ but H and $\mathrm{Spin}(9)$ have different Lie types. Hence H has type A_4. The morphism $H \hookrightarrow \mathrm{Spin}(9) \to \mathrm{SO}(9)$ gives H a self-dual 9-dimensional representation of H, which leads to contradiction since the A_4-type group H does not admit such a representation. □

4.5.1 Cases except A_2

In the remaining possible Lie types for connected simple subgroups of F_4, the type A_2 is more complicated. So we first look at the other types:

Proposition 4.5.3. *(1) For each type among*

$$\mathrm{A}_3, \mathrm{B}_2, \mathrm{B}_3, \mathrm{B}_4, \mathrm{C}_3, \mathrm{D}_4, \mathrm{G}_2,$$

there exists a simply-connected subgroup of F_4 *with this type.*
(2) Let H *be a connected compact Lie group such that it admits an embedding into* F_4 *and its Lie type is among*

$$\mathrm{A}_3, \mathrm{B}_2, \mathrm{B}_3, \mathrm{B}_4, \mathrm{C}_3, \mathrm{D}_4, \mathrm{G}_2.$$

Then H *is simply-connected and the embedding* $H \hookrightarrow \mathrm{F}_4$ *is unique up to conjugacy.*

Before proving this proposition case by case, we explain our strategy. Fixing a Lie type, we first construct an embedding ϕ_0 from the simply-connected compact Lie group H_0 of the given type into F_4. We claim that to prove Proposition 4.5.3(2) for this Lie type, it suffices to show that for any connected simple compact Lie group H of the same type with H_0, i.e. H is isomorphic to the quotient of H_0 by a finite central subgroup, and any embedding $\phi : H \to \mathrm{F}_4$, the restriction of the 26-dimensional irreducible representation J_0 along ϕ is unique, up to equivalence of H_0-representations. Here we view the restriction of J_0 along $\phi : H \to \mathrm{F}_4$ as a representation of H_0 by the composition with a central isogeny $H_0 \to H$.

Proof of the claim. For a connected compact Lie group H of the same Lie type as H_0 and an embedding $\phi : H \hookrightarrow \mathrm{F}_4$, we can lift ϕ to a morphism $\phi \circ i : H_0 \to \mathrm{F}_4$ via a central isogeny $i : H_0 \to H$. This morphism $\phi \circ i$ is conjugate to ϕ_0 by the uniqueness of $\mathrm{J}_0|_{H_0}$ and Proposition 4.2.1, thus i is injective, which implies that H is also simply-connected. For any two embeddings $\phi, \phi' : H \hookrightarrow \mathrm{F}_4$, applying Proposition 4.2.1 to $\phi \circ i$ and $\phi' \circ i$, we have $\phi \circ i$ and $\phi' \circ i$ are conjugate in F_4, thus ϕ and ϕ' are conjugate. □

4.5.1.1 B_4 In this case $H_0 \simeq \mathrm{Spin}(9)$ and we take ϕ_0 to be $H_0 \simeq \mathrm{Spin}(9) \hookrightarrow \mathrm{F}_4$, where $\mathrm{Spin}(9) \hookrightarrow \mathrm{F}_4$ is constructed in §4.3.1.

For any embedding ϕ from a B_4-type connected compact Lie group H into F_4, by Lemma 4.5.1 the image $\mathrm{Im}(\phi)$ (up to conjugate) is a subgroup of the $\mathrm{Spin}(9)$ in F_4, thus ϕ factors through an embedding $H \to \mathrm{Spin}(9)$. This embedding must be an isomorphism, so the restrictions of J_0 along ϕ_0 and ϕ are equivalent as H_0-representations.

4.5.1.2 D_4 In this case $H_0 \simeq \mathrm{Spin}(8)$ and we take ϕ_0 to be the composition of the natural embedding $\mathrm{Spin}_8 \hookrightarrow \mathrm{Spin}(9)$ with $\mathrm{Spin}(9) \hookrightarrow \mathrm{F}_4$.

For any embedding ϕ from a D_4-type connected compact Lie group H into F_4, by Lemma 4.5.1, ϕ (up to conjugacy) factors through an embedding $H \to \mathrm{Spin}(9)$. The restriction of the 9-dimensional irreducible representation V_9 to H is isomorphic to either $\mathbf{1} + \mathrm{V}_8$ or $\mathbf{1} + \mathrm{V}_{\mathrm{Spin}}^{+}$ or $\mathbf{1} + \mathrm{V}_{\mathrm{Spin}}^{-}$, where V_8 is the standard 8-dimensional representation of $\mathrm{Spin}(8)$, and $\mathrm{V}_{\mathrm{Spin}}^{\pm}$ are two 8-dimensional spinor representations of $\mathrm{Spin}(8)$. For those three possibilities, we obtain the same equivalence class of $\mathrm{J}_0|_H$, which is equivalent to $\mathbf{1}^{\oplus 2} + \mathrm{V}_8 + \mathrm{V}_{\mathrm{Spin}}^{+} + \mathrm{V}_{\mathrm{Spin}}^{-}$ as H_0-representations. This representation is stable under the outer automorphisms of H_0, so the restriction of J_0 along ϕ is unique, up to equivalence of H_0-representations.

4.5.1.3 A_3 In this case $H_0 \simeq \mathrm{SU}(4)$, and we take ϕ_0 to be the composition of the natural embedding $\mathrm{SU}(4) \simeq \mathrm{Spin}(6) \hookrightarrow \mathrm{Spin}(9)$ with $\mathrm{Spin}(9) \hookrightarrow \mathrm{F}_4$.

For any embedding ϕ from a A_3-type connected compact Lie group H into F_4, by Lemma 4.5.1, ϕ (up to conjugacy) factors through an embedding from H to $\mathrm{Sp}(3)$ or $\mathrm{Spin}(9)$.

If ϕ factors through $\mathrm{Sp}(3)$, then the image of ϕ gives a A_3-type subgroup of $\mathrm{Sp}(3)$. This subgroup of $\mathrm{Sp}(3)$ must be regular, but this contradicts with the Borel-de Siebenthal theory.

If ϕ factors through $\mathrm{Spin}(9)$, the standard representation V_9 of $\mathrm{Spin}(9)$ gives a self-dual 9-dimensional representation of H. Up to equivalence, there are two possibilities for the restriction of V_9 to H:

$$\mathbf{1}^{\oplus 3} + \wedge^2 \mathrm{V}_4 \text{ or } \mathbf{1} + \mathrm{V}_4 + \mathrm{V}_4',$$

where V_4 is the standard 4-dimensional representation of $\mathrm{SU}(4)$ and V_4' is its dual. For both cases, the restriction of the irreducible representation J_0 of F_4 along ϕ is isomorphic to

$$\mathbf{1}^{\oplus 4} + \mathrm{V}_4^{\oplus 2} + (\mathrm{V}_4')^{\oplus 2} + \wedge^2 \mathrm{V}_4.$$

This representation is stable under the outer automorphism of H_0, so the restriction of J_0 along ϕ is unique, up to equivalence of H_0-representations.

4.5.1.4 B_3 In this case $H_0 \simeq \mathrm{Spin}(7)$, and we take ϕ_0 to be the composition of the natural embedding $\mathrm{Spin}(7) \hookrightarrow \mathrm{Spin}(9)$ with $\mathrm{Spin}(9) \hookrightarrow \mathrm{F}_4$.

For any embedding ϕ from a B_3-type connected compact Lie group H into F_4, by Lemma 4.5.1 and the Borel-de Siebenthal theory, ϕ (up to conjugacy) factors through an embedding from H to $\mathrm{Spin}(9)$. The restriction of the standard representation V_9 of $\mathrm{Spin}(9)$ to H must be isomorphic to either $\mathbf{1}^{\oplus 2} + \mathrm{V}_7$ or $\mathbf{1} + \mathrm{V}_{\mathrm{Spin}}$, where V_7 is the standard 7-dimensional representation of $\mathrm{Spin}(7)$, and $\mathrm{V}_{\mathrm{Spin}}$ is the 8-dimensional spinor representation of $\mathrm{Spin}(7)$. For both cases, the restriction of the irreducible representation J_0 of F_4 along ϕ is isomorphic to

$$\mathbf{1}^{\oplus 3} + \mathrm{V}_7 + \mathrm{V}_{\mathrm{Spin}}^{\oplus 2}.$$

Hence the restriction of J_0 along ϕ is unique, up to equivalence of H_0-representations.

4.5.1.5 C_3 In this case $H_0 \simeq \mathrm{Sp}(3)$, and we take ϕ_0 to be $\mathrm{Sp}(3) \hookrightarrow (\mathrm{Sp}(1) \times \mathrm{Sp}(3))/\mu_2^{\Delta} \hookrightarrow \mathrm{F}_4$, where the subgroup $(\mathrm{Sp}(1) \times \mathrm{Sp}(3))/\mu_2^{\Delta}$ is given in §4.3.2.

For any embedding ϕ from a C_3-type connected compact Lie group H into F_4, by Lemma 4.5.1, ϕ (up to conjugacy) factors through a central-kernel morphism from H_0 to $\mathrm{Sp}(3)$ or $\mathrm{Spin}(9)$.

If ϕ factors through $\mathrm{Spin}(9)$, then the standard representation V_9 of $\mathrm{Spin}(9)$ induces an orthogonal 9-dimensional representation of $\mathrm{Sp}(3)$. However, each non-trivial irreducible orthogonal representation of $\mathrm{Sp}(3)$ has dimension larger than 9, which leads to a contradiction.

If ϕ factors through $\mathrm{Sp}(3)$, then the embedding $H \to \mathrm{Sp}(3)$ must be an isomorphism. This implies that the restriction of the irreducible representation J_0 of F_4 along ϕ is isomorphic to $\mathrm{V}_6^{\oplus 2} + \mathrm{V}_{14}$, where V_6 and V_{14} stand for the same representations in (4.3). Hence the restriction of J_0 along ϕ is unique, up to equivalence of H_0-representations.

4.5.1.6 B_2 In this case $H_0 \simeq \mathrm{Sp}(2) \simeq \mathrm{Spin}(5)$, and we take ϕ_0 to be the composition of the natural embedding $\mathrm{Sp}(2) \hookrightarrow \mathrm{Sp}(3) \hookrightarrow (\mathrm{Sp}(1) \times \mathrm{Sp}(3))/\mu_2^{\Delta}$ with the embedding $(\mathrm{Sp}(1) \times \mathrm{Sp}(3))/\mu_2^{\Delta} \hookrightarrow \mathrm{F}_4$ given in §4.3.2.

For any embedding ϕ from a B_2-type connected compact Lie group H into F_4, by Lemma 4.5.1 and the Borel-de Siebenthal theory, ϕ (up to conjugacy) factors through an embedding from H to $\mathrm{Sp}(3)$ or $\mathrm{Spin}(9)$.

If ϕ factors through $\mathrm{Sp}(3)$, then the restriction of the standard representation V_6 of $\mathrm{Sp}(3)$ to H must be isomorphic to $\mathbf{1}^{\oplus 2} + \mathrm{V}_4$, where V_4 is the standard 4-dimensional symplectic representation of $\mathrm{Sp}(2)$. The restriction of the irreducible representation J_0 along ϕ is isomorphic to $\mathbf{1}^{\oplus 5} + \mathrm{V}_4^{\oplus 4} + \mathrm{V}_5$, where V_5 is the standard 5-dimensional orthogonal representation of $\mathrm{Spin}(5)$.

If ϕ factors through $\mathrm{Spin}(9)$, then the restriction of the standard representation V_9 to H must be isomorphic to $\mathbf{1}^{\oplus 4} + \mathrm{V}_5$ or $\mathbf{1} + \mathrm{V}_4^{\oplus 2}$. For these two possibilities, the restriction of J_0 along ϕ is isomorphic to $\mathbf{1}^{\oplus 5} + \mathrm{V}_4^{\oplus 4} + \mathrm{V}_5$. Hence the restriction of J_0 along ϕ is unique, up to equivalence of H_0-representations.

4.5.1.7 G_2 In this case $H_0 \simeq \mathrm{G}_2$, and we take ϕ_0 to be the embedding $\mathrm{G}_2 \hookrightarrow \mathrm{G}_2 \times \mathrm{SO}(3) \hookrightarrow \mathrm{F}_4$, as given in §4.3.4.

Combining Lemma 4.5.1 and the fact that all non-trivial representations of G_2 have dimension larger than 6, any embedding ϕ from a G_2-type connected compact Lie group H into F_4 (up to conjugacy) factors through an embedding from H to $\mathrm{Spin}(9)$. The restriction of the standard representation V_9 of $\mathrm{Spin}(9)$ to H must be isomorphic to $\mathbf{1}^{\oplus 2} + \mathrm{V}_7$, where V_7 is the same as in (4.10). So the restriction of the representation J_0 of F_4 along ϕ must be isomorphic to $\mathbf{1}^{\oplus 5} + \mathrm{V}_7^{\oplus 3}$. Hence the restriction of J_0 along ϕ is unique, up to equivalence of H_0-representations.

4.5.2 The case A_2

For the Lie type A_2, our idea is the same with the proof of Proposition 4.5.3, but this time we have several conjugacy classes of embeddings from a A_2-type group to F_4.

Proposition 4.5.4. *(1) There are* 3 *conjugacy classes of embeddings from* $\mathrm{SU}(3)$ *to* F_4*,*
(2) There is a unique conjugacy class of embeddings from $\mathrm{PSU}(3) = \mathrm{SU}(3)/\mathrm{Z}(\mathrm{SU}(3))$ *to* F_4*.*

Proof. By Lemma 4.5.1, any embedding ϕ from a connected A_2-type compact Lie group H to F_4 (up to conjugacy) factors through $\mathrm{Spin}(9)$ or $\mathrm{Sp}(3)$ or $(\mathrm{SU}(3) \times \mathrm{SU}(3))/\mu_2^{\Delta}$.

We start from the case that ϕ factors through $(\mathrm{SU}(3) \times \mathrm{SU}(3))/\mu_3^{\Delta}$. Fix an embedding $\iota : (\mathrm{SU}(3) \times \mathrm{SU}(3))/\mu_3^{\Delta} \hookrightarrow \mathrm{F}_4$ such that the restriction of the irreducible representation J_0 of F_4 along this embedding is isomorphic to (4.6). We denote the outer automorphism of $\mathrm{SU}(3)$ by θ. It is easy to classify the conjugacy classes of embeddings $\psi : H \hookrightarrow (\mathrm{SU}(3) \times \mathrm{SU}(3))/\mu_3^{\Delta}$, where H is a connected A_2-type compact Lie group, i.e. $H \simeq \mathrm{SU}(3)$ or $\mathrm{PSU}(3)$. We list the conjugacy classes as follows:

Index	H	ψ	The restriction of J_0 along $\phi = \iota \circ \psi$
1	$\mathrm{SU}(3)$	$g \mapsto (g, 1)$	$(\mathrm{V}_3 + \mathrm{V}_3')^{\oplus 3} + \mathfrak{sl}_3$
2	$\mathrm{SU}(3)$	$g \mapsto (1, g)$	$\mathbf{1}^{\oplus 8} + (\mathrm{V}_3 + \mathrm{V}_3')^{\oplus 3}$
3	$\mathrm{PSU}(3)$	$g \mapsto (g, g)$	$\mathbf{1}^{\oplus 2} + \mathfrak{sl}_3^{\oplus 3}$
4	$\mathrm{SU}(3)$	$g \mapsto (g, \theta(g))$	$\mathrm{V}_3 + \mathrm{V}_3' + \mathrm{Sym}^2\,\mathrm{V}_3 + \mathrm{Sym}^2\,\mathrm{V}_3' + \mathfrak{sl}_3$

Table 3: Embeddings from A_2-type connected compact Lie groups to $(\mathrm{SU}(3) \times \mathrm{SU}(3))/\mu_3^{\Delta}$

The representations of $\mathrm{SU}(3)$ appearing in this table have been explained in §4.3.3. If we choose the embedding ι to be the one corresponding to (4.5), then by Proposition 4.2.1 we get the same conjugacy classes of embeddings.

If ϕ factors through $\mathrm{Sp}(3)$, the standard representation V_6 of $\mathrm{Sp}(3)$ gives a self-dual 6-dimensional representation of H, thus the restriction of V_6 to H must be isomorphic to $\mathrm{V}_3 + \mathrm{V}_3'$. So the restriction of J_0 to H is isomorphic to $(\mathrm{V}_3 + \mathrm{V}_3')^{\oplus 3} + \mathfrak{sl}_3$.

If ϕ factors through $\mathrm{Spin}(9)$, the standard representation V_9 of $\mathrm{Spin}(9)$ gives a self-dual 9-dimensional representation of H, thus the restriction of V_9 to H must be isomorphic to $\mathbf{1}^{\oplus 3} + \mathrm{V}_3 + \mathrm{V}_3'$ or $\mathbf{1} + \mathfrak{sl}_3$. For the first case, the restriction of J_0 to H is isomorphic to $\mathbf{1}^{\oplus 8} + (\mathrm{V}_3 + \mathrm{V}_3')^{\oplus 3}$, and for the second case, the restriction of J_0 to H is isomorphic to $\mathbf{1}^{\oplus 2} + \mathfrak{sl}_3^{\oplus 3}$.

In conclusion, combining Proposition 4.2.1 with our analysis on the restriction of J_0, we get that every embedding from a connected A_2-type compact Lie group to F_4 is conjugate to one of the embeddings $\phi = \iota \circ \psi$ in Table 3. □

4.5.3 Centralizers

Similarly with the arguments in §4.4.5, using Lemma 4.4.7 and Lemma 4.4.8, for each conjugacy class of embeddings from a connected simple compact Lie group to F_4, we can determine its centralizer in F_4:

- Type B_4: the centralizer is a cyclic group of order 2.
- Type D_4: the centralizer is isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.
- Type A_3: the centralizer is an A_1-subgroup in the class $[3, 2^8, 1^7]$, which is labeled by $\widetilde{\mathrm{A}_1}$.
- Type B_3: the centralizer is the product of a rank 1 torus with a cyclic group of order 2.
- Type C_3: the centralizer is an A_1-subgroup in the class $[2^6, 1^{14}]$, which is labeled by A_1.
- Type B_2: the centralizer is the direct product of two A_1-subgroups in the class $[2^6, 1^{14}]$.
- Type G_2: the centralizer is an A_1-subgroup in the class $[5, 3^7]$, which is labeled by $\widetilde{\mathrm{A}_2}$.
- Type A_2: Let $\phi : H \hookrightarrow \mathrm{F}_4$ be a representative of a conjugacy class of embeddings listed in Table 3, which is indexed by a number from 1 to 4.

  (1) If ϕ is indexed by 1, then its centralizer is conjugate to the $\mathrm{SU}(3)$ indexed by 2.
  (2) If ϕ is indexed by 2, then its centralizer is conjugate to the $\mathrm{SU}(3)$ indexed by 1.
  (3) If ϕ is indexed by 3, then its centralizer is finite and contains an order 3 element.

(4) If ϕ is indexed by 4, then its centralizer is a cyclic group of order 3.

4.6 Connected subgroups satisfying certain conditions

After a long journey of classifying conjugacy classes of connected simple subgroups of F_4 and computing their centralizers in F_4, we are finally able to enumerate all the connected subgroups H of F_4 satisfying our three conditions listed in the beginning of §4.

We first classify all the connected subgroups H of F_4 such that $\mathrm{C}_{\mathrm{F}_4}(H)$ is an elementary finite abelian 2-group, via our classifications in §4.4 and §4.5.

Notation 4.6.1. From now on, for an A_1-subgroup of F_4, if its conjugacy class corresponds to the partition p of 26, we will simply denote this A_1-subgroup by A_1^p. For example, we will denote the principal $\mathrm{PSU}(2)$ of F_4 by $\mathrm{A}_1^{[17,9]}$. For an A_2-type subgroup of F_4, if its conjugacy class is indexed by $n \in \{1,2,3,4\}$ in §4.5.2 Table 3, then we denote it simply by $\mathrm{A}_2^{(n)}$.

Now let H be a connected subgroup of F_4 whose centralizer in F_4 is an elementary finite abelian 2-group. Let Φ be the root system of H, and we can write it as a disjoint union of irreducible root systems:

$$\Phi = \Phi_1 \sqcup \cdots \sqcup \Phi_s.$$

We denote by m the number of $i \in \{1,2,\ldots,s\}$ such that $\Phi_i \simeq \mathrm{A}_1$.

Lemma 4.6.2. *If $s = 1$, i.e. H is simple, then H is conjugate to one of the following subgroups of F_4:*

$$\mathrm{F}_4, \mathrm{Spin}(9), \mathrm{Spin}(8), \mathrm{A}_1^{[17,9]}, \mathrm{A}_1^{[11,9,5,1]}, \mathrm{A}_1^{[9,7,5^2]}.$$

Proof. By our computations in §4.4.5 and §4.5.3, we have if the centralizer of H in F_4 is finite, then it must be conjugate to one of the following subgroups of F_4:

$$\mathrm{F}_4, \mathrm{Spin}(9), \mathrm{Spin}(8), \mathrm{A}_2^{(3)}, \mathrm{A}_2^{(4)}, \mathrm{A}_1^{[17,9]}, \mathrm{A}_1^{[11,9,5,1]}, \mathrm{A}_1^{[9,7,5^2]}, \mathrm{A}_1^{[5^3,3^3,1^2]}.$$

According to § 4.4.5.5 and §4.5.3, if H is in the conjugacy class of $\mathrm{A}_2^{(3)}, \mathrm{A}_2^{(4)}$ or $\mathrm{A}_1^{[5^3,3^3,1^2]}$, then the centralizer of H in F_4 contains an element of order 3. $\square$

Lemma 4.6.3. *If $s > 1$ and $m = 0$, then there is no such H satisfying $\mathrm{C}_{\mathrm{F}_4}(H)$ is an elementary finite abelian 2-group.*

Proof. Since $s > 1$ and $m = 0$, the irreducible root systems Φ_1 and Φ_2 both have rank 2 and $s = 2$. Hence H must be isomorphic to the quotient of $\mathrm{SU}(3)\times\mathrm{SU}(3)$ by a finite central subgroup. By our classification in §4.5.2, H is conjugate to the subgroup $(\mathrm{SU}(3)\times\mathrm{SU}(3))/\mu_3^{\Delta}$ constructed in §4.3.3. However, the centralizer of this subgroup contains its center, which is a cyclic group of order 3, so in this case there is no H whose centralizer in F_4 is an elementary finite abelian 2-group. $\square$

Lemma 4.6.4. *If $s = 2$ and $m \geq 1$, then H is conjugate to one of the following subgroups of F_4:*

$$\left(\mathrm{A}_1^{[2^6,1^{14}]} \times \mathrm{Sp}(3)\right)/\mu_2^{\Delta}, \left(\mathrm{A}_1^{[3,2^8,1^7]} \times \mathrm{Spin}(5)\right)/\mu_2^{\Delta}, \mathrm{A}_1^{[5,3^7]} \times \mathrm{G}_2,$$
$$\mathrm{A}_1^{[7^3,1^5]} \times \mathrm{A}_1^{[5,3^7]}, \left(\mathrm{A}_1^{[9,6^2,5]} \times \mathrm{A}_1^{[2^6,1^{14}]}\right)/\mu_2^{\Delta}, \left(\mathrm{A}_1^{[5^2,4^2,3,2^2,1]} \times \mathrm{A}_1^{[2^6,1^{14}]}\right)/\mu_2^{\Delta},$$
$$\left(\mathrm{A}_1^{[5,4^4,1^5]} \times \mathrm{A}_1^{[3,2^8,1^7]}\right)/\mu_2^{\Delta}, \left(\mathrm{A}_1^{[5,4^2,3^3,2^2]} \times \mathrm{A}_1^{[3^3,2^6,1^5]}\right)/\mu_2^{\Delta}, \left(\mathrm{A}_1^{[4^2,3^3,2^4,1]} \times \mathrm{A}_1^{[4^2,3^3,2^4,1]}\right)/\mu_2^{\Delta}.$$

Proof. Since $s = 2$ and $m \geq 1$, up to conjugacy H is of the form $(X \times H_0)/\Gamma$, where X is an A_1-subgroup of F_4, H_0 is a connected simple subgroup of F_4, and Γ is either trivial or the subgroup μ_2^Δ of $X \times H_0$. Since the centralizer of H in F_4 is an elementary finite abelian 2-groups, the centralizer of H_0 in $\mathrm{C}_{\mathrm{F}_4}(X)$ and the centralizer of X in $\mathrm{C}_{\mathrm{F}_4}(X)$ are both elementary finite abelian 2-groups.

If $\mathrm{rank}(H_0) > 1$, by §4.5.3 we have the following possibilities for the conjugacy class of H:

$$\left(\mathrm{A}_1^{[2^6,1^{14}]} \times \mathrm{Sp}(3)\right)/\mu_2^\Delta, \left(\mathrm{A}_1^{[3,2^8,1^7]} \times \mathrm{Spin}(5)\right)/\mu_2^\Delta, \mathrm{A}_1^{[5,3^7]} \times \mathrm{G}_2.$$

If H_0 is also an A_1-subgroup of F_4, by §4.4.5 we have the following possibilities for the conjugacy class of H:

$$\mathrm{A}_1^{[7^3,1^5]} \times \mathrm{A}_1^{[5,3^7]}, \left(\mathrm{A}_1^{[9,6^2,5]} \times \mathrm{A}_1^{[2^6,1^{14}]}\right)/\mu_2^\Delta, \left(\mathrm{A}_1^{[5^2,4^2,3,2^2,1]} \times \mathrm{A}_1^{[2^6,1^{14}]}\right)/\mu_2^\Delta,$$
$$\left(\mathrm{A}_1^{[5,4^4,1^5]} \times \mathrm{A}_1^{[3,2^8,1^7]}\right)/\mu_2^\Delta, \left(\mathrm{A}_1^{[5,4^2,3^3,2^2]} \times \mathrm{A}_1^{[3^3,2^6,1^5]}\right)/\mu_2^\Delta, \left(\mathrm{A}_1^{[4^2,3^3,2^4,1]} \times \mathrm{A}_1^{[4^2,3^3,2^4,1]}\right)/\mu_2^\Delta.$$

□

Lemma 4.6.5. *If $s > 2$, then H is conjugate to one of the following subgroups of F_4:*

$$\left(\mathrm{A}_1^{[2^6,1^{14}]} \times \mathrm{A}_1^{[2^6,1^{14}]} \times \mathrm{Sp}(2)\right)/\mu_2^\Delta,$$
$$\mathrm{A}_1^{[5,3^7]} \times \left(\mathrm{A}_1^{[3^3,2^6,1^5]} \times \mathrm{A}_1^{[2^6,1^{14}]}\right)/\mu_2^\Delta,$$
$$\left(\mathrm{A}_1^{[5,4^4,1^5]} \times \mathrm{A}_1^{[2^6,1^{14}]} \times \mathrm{A}_1^{[2^6,1^{14}]}\right)/\mu_2^\Delta,$$
$$\left(\mathrm{A}_1^{[3,2^8,1^7]} \times \mathrm{A}_1^{[3,2^8,1^7]} \times \mathrm{A}_1^{[3,2^8,1^7]}\right)/\langle(1,-1,-1),(-1,-1,1)\rangle,$$
$$\prod_{i=1}^4 \mathrm{A}_1^{[2^6,1^{14}]}/\mu_2^\Delta := \left(\mathrm{A}_1^{[2^6,1^{14}]} \times \mathrm{A}_1^{[2^6,1^{14}]} \times \mathrm{A}_1^{[2^6,1^{14}]} \times \mathrm{A}_1^{[2^6,1^{14}]}\right)/\mu_2^\Delta.$$

Proof. This follows from a similar argument as in the proof of Lemma 4.6.4 and the results in §4.4.5 and §4.5.3. □

In Lemma 4.6.2, Lemma 4.6.3, Lemma 4.6.4 and Lemma 4.6.5, we have enumerated all the conjugacy classes of connected subgroups H of F_4 such that the centralizer of H in F_4 is an elementary finite abelian 2-group. There are 20 such conjugacy classes, but some of them do not satisfy the third condition given in the beginning of §4:

Lemma 4.6.6. *If a subgroup H of F_4 is conjugate to one of the following subgroups:*

$$\mathrm{A}_1^{[11,9,5,1]}, \mathrm{A}_1^{[9,7,5^2]}, \left(\mathrm{A}_1^{[3,2^8,1^7]} \times \mathrm{Spin}(5)\right)/\mu_2^\Delta, \left(\mathrm{A}_1^{[5^2,4^2,3,2^2,1]} \times \mathrm{A}_1^{[2^6,1^{14}]}\right)/\mu_2^\Delta,$$
$$\left(\mathrm{A}_1^{[5,4^4,1^5]} \times \mathrm{A}_1^{[3,2^8,1^7]}\right)/\mu_2^\Delta, \mathrm{A}_1^{[3,2^8,1^7]} \times \mathrm{A}_1^{[3,2^8,1^7]} \times \mathrm{A}_1^{[3,2^8,1^7]}/\langle(1,-1,-1),(-1,-1,1)\rangle,$$

then the zero weight appears 4 times in the restriction of the 26-dimensional irreducible representation J_0 of F_4 to H.

Proof. The restrictions of the representation J_0 of F_4 to the two A_1-subgroups in the list above can be read from their corresponding partitions. In both cases, the multiplicity of the zero weight in $\mathrm{J}_0|_H$ is 4.

If H is conjugate to $\left(\mathrm{A}_1^{[3,2^8,1^7]} \times \mathrm{Spin}(5)\right)/\mu_2^{\Delta}$, then the restriction $\mathrm{J}_0|_H$ is isomorphic to

$$\left(\mathbf{1}^{\oplus 2} + \mathrm{Sym}^2\,\mathrm{St}\right) \otimes \mathbf{1} + \mathrm{St}^{\oplus 2} \otimes \mathrm{V}_4 + \mathbf{1} \otimes \mathrm{V}_5,$$

in which the zero weight appears 4 times.

If H is conjugate to $\left(\mathrm{A}_1^{[5^2,4^2,3,2^2,1]} \times \mathrm{A}_1^{[2^6,1^{14}]}\right)/\mu_2^{\Delta}$, then the restriction $\mathrm{J}_0|_H$ is isomorphic to

$$\left((\mathrm{Sym}^4\,\mathrm{St})^{\oplus 2} + \mathrm{Sym}^2\,\mathrm{St} + \mathbf{1}\right) \otimes \mathbf{1} + \left(\mathrm{Sym}^3\,\mathrm{St} + \mathrm{St}\right) \otimes \mathrm{St},$$

in which the zero weight appears 4 times.

If H is conjugate to $\left(\mathrm{A}_1^{[5,4^4,1^5]} \times \mathrm{A}_1^{[3,2^8,1^7]}\right)/\mu_2^{\Delta}$, then the restriction $\mathrm{J}_0|_H$ is isomorphic to

$$\mathbf{1} \otimes \left(\mathbf{1}^{\oplus 2} + \mathrm{Sym}^2\,\mathrm{St}\right) + \left(\mathrm{Sym}^3\,\mathrm{St} \otimes \mathrm{St}\right)^{\oplus 2} + \mathrm{Sym}^4\,\mathrm{St} \otimes \mathbf{1},$$

in which the zero weight appears 4 times.

If H is conjugate to $\mathrm{A}_1^{[3,2^8,1^7]} \times \mathrm{A}_1^{[3,2^8,1^7]} \times \mathrm{A}_1^{[3,2^8,1^7]}/\langle(1,-1,-1),(-1,-1,1)\rangle$, then the restriction $\mathrm{J}_0|_H$ is isomorphic to

$$\mathbf{1} + (\mathrm{St} \otimes \mathrm{St} \otimes \mathrm{St})^{\oplus 2} + \mathrm{Sym}^2\,\mathrm{St} \otimes \mathbf{1} \otimes \mathbf{1} + \mathbf{1} \otimes \mathrm{Sym}^2\,\mathrm{St} \otimes \mathbf{1} + \mathbf{1} \otimes \mathbf{1} \otimes \mathrm{Sym}^2\,\mathrm{St},$$

in which the zero weight appears 4 times. $\square$

In conclusion, we have proved the following theorem:

Theorem 4.6.7. *There are* 13 *conjugacy classes of proper connected subgroups H of F_4 satisfying the following conditions:*

(1) The centralizer of H in F_4 is an elementary finite abelian 2-group.
(2) The zero weight appears twice in the restriction of the 26-dimensional irreducible representation J_0 of F_4 to H.

These 13 *subgroups are:*

$$\mathrm{A}_1^{[17,9]}, \mathrm{Spin}(9), \mathrm{Spin}(8), \mathrm{A}_1^{[5,3^7]} \times \mathrm{G}_2, \mathrm{A}_1^{[7^3,1^5]} \times \mathrm{A}_1^{[5,3^7]}, \left(\mathrm{A}_1^{[2^6,1^{14}]} \times \mathrm{Sp}(3)\right)/\mu_2^{\Delta},$$
$$\left(\mathrm{A}_1^{[2^6,1^{14}]} \times \mathrm{A}_1^{[2^6,1^{14}]} \times \mathrm{Sp}(2)\right)/\mu_2^{\Delta}, \left(\mathrm{A}_1^{9,6^2,5} \times \mathrm{A}_1^{[2^6,1^{14}]}\right)/\mu_2^{\Delta}, \left(\mathrm{A}_1^{[5,4^2,3^3,2^2]} \times \mathrm{A}_1^{[3^3,2^6,1^5]}\right)/\mu_2^{\Delta},$$
$$\left(\mathrm{A}_1^{[4^2,3^3,2^4,1]} \times \mathrm{A}_1^{[4^2,3^3,2^4,1]}\right)/\mu_2^{\Delta}, \mathrm{A}_1^{[5,3^7]} \times \left(\mathrm{A}_1^{[3^3,2^6,1^5]} \times \mathrm{A}_1^{[2^6,1^{14}]}\right)/\mu_2^{\Delta},$$
$$\left(\mathrm{A}_1^{[5,4^4,1^5]} \times \mathrm{A}_1^{[2^6,1^{14}]} \times \mathrm{A}_1^{[2^6,1^{14}]}\right)/\mu_2^{\Delta}, \prod_{i=1}^{4} \mathrm{A}_1^{[2^6,1^{14}]}/\mu_2^{\Delta}.$$

For the 13 conjugacy classes of subgroups H in Theorem 4.6.7, in the rest of this subsection we are going to list some information will be used in §6:

- the centralizer $\mathrm{C}_{\mathrm{F}_4}(H)$ of H in F_4,
- the restriction of the 26-dimensional irreducible representation J_0 to H,
- and the restriction of the adjoint representation $\mathfrak{f}_4$ of F_4 to H.

4.6.1 $\mathrm{A}_1^{[17,9]}$

This is the principal $\mathrm{PSU}(2)$ of F_4, whose centralizer in F_4 is trivial. The restriction of J_0 to H corresponds to the partition $[17,9]$ of 26, and the restriction of $\mathfrak{f}_4$ to H corresponds to the partition $[23,15,11,3]$ of 52.

4.6.2 $\mathrm{Spin}(9)$

The centralizer of H in F_4 is the center of H, which is isomorphic to $\mathbb{Z}/2\mathbb{Z}$.
The restriction of J_0 to H is isomorphic to

$$\mathbf{1} + \mathrm{V}_9 + \mathrm{V}_{\mathrm{Spin}},$$

and the restriction of $\mathfrak{f}_4$ to H is isomorphic to

$$\wedge^2 \mathrm{V}_9 + \mathrm{V}_{\mathrm{Spin}},$$

where V_9 is the standard representation of $\mathrm{Spin}(9)$ and $\mathrm{V}_{\mathrm{Spin}}$ is the 16-dimensional spinor representation.

4.6.3 $\left(\mathrm{A}_1^{[2^6,1^{14}]} \times \mathrm{Sp}(3)\right)/\mu_2^{\Delta}$

The centralizer of H in F_4 is the center of H, which is isomorphic to $\mathbb{Z}/2\mathbb{Z}$.
The restriction of J_0 to H is isomorphic to

$$\mathrm{St} \otimes \mathrm{V}_6 + \mathbf{1} \otimes \mathrm{V}_{14},$$

and the restriction of $\mathfrak{f}_4$ to H is isomorphic to

$$\mathrm{Sym}^2 \mathrm{St} \otimes \mathbf{1} + \mathrm{St} \otimes \mathrm{V}'_{14} + \mathbf{1} \otimes \mathrm{Sym}^2 \mathrm{V}_6,$$

where V_6 is the standard 6-dimensional representation of $\mathrm{Sp}(3)$, V_{14} is the 14-dimensional irreducible representation of $\mathrm{Sp}(3)$ that is a sub-representation of $\wedge^2\mathrm{V}_6$, and V'_{14} is another 14-dimensional irreducible representation of $\mathrm{Sp}(3)$ that is not equivalent to V_{14}. From now on, we will denote V_{14} by $\wedge^*\mathrm{V}_6$, and similarly for the 5-dimensional irreducible representation of $\mathrm{Sp}(2)$.

4.6.4 $\mathrm{A}_1^{[5,3^7]} \times \mathrm{G}_2$

The centralizer of H in F_4 is trivial.
The restriction of J_0 to H is isomorphic to

$$\mathrm{Sym}^2 \mathrm{St} \otimes \mathrm{V}_7 + \mathrm{Sym}^4 \mathrm{St} \otimes \mathbf{1},$$

and the restriction of $\mathfrak{f}_4$ to this subgroup is isomorphic to

$$\mathbf{1} \otimes \mathfrak{g}_2 + \mathrm{Sym}^2 \mathrm{St} \otimes \mathbf{1} + \mathrm{Sym}^4 \mathrm{St} \otimes \mathrm{V}_7,$$

where V_7 is the 7-dimensional irreducible representation of G_2, and $\mathfrak{g}_2$ is the adjoint representation of G_2.

4.6.5 Spin(8)

The centralizer of H in F_4 is the center of H, which is isomorphic to $\mathrm{Z}(\mathrm{Spin}(8)) \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$.

The restriction of J_0 to H is isomorphic to

$$\mathbf{1}^{\oplus 2} + \mathrm{V}_8 + \mathrm{V}_{\mathrm{Spin}}^{+} + \mathrm{V}_{\mathrm{Spin}}^{-},$$

and the restriction of $\mathfrak{f}_4$ to H is isomorphic to

$$\wedge^2 \mathrm{V}_8 + \mathrm{V}_8 + \mathrm{V}_{\mathrm{Spin}}^{+} + \mathrm{V}_{\mathrm{Spin}}^{-},$$

where V_8 is the 8-dimensional vector representation of Spin(8), i.e. the composition of $\mathrm{Spin}(8) \to \mathrm{SO}(8)$ with the standard 8-dimensional representation of SO(8), and $\mathrm{V}_{\mathrm{Spin}}^{\pm}$ are two 8-dimensional spinor representations.

4.6.6 $\left(\mathrm{A}_1^{[2^6,1^{14}]} \times \mathrm{A}_1^{[2^6,1^{14}]} \times \mathrm{Sp}(2)\right)/\mu_2^{\Delta}$

The centralizer of H in F_4 is the center of H, which is isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

The restriction of J_0 to H is isomorphic to

$$\mathbf{1} + \mathrm{St} \otimes \mathrm{St} \otimes \mathbf{1} + \mathrm{St} \otimes \mathbf{1} \otimes \mathrm{V}_4 + \mathbf{1} \otimes \mathrm{St} \otimes \mathrm{V}_4 + \mathbf{1} \otimes \mathbf{1} \otimes \wedge^* \mathrm{V}_4,$$

and the restriction of $\mathfrak{f}_4$ to H is isomorphic to

$$\begin{gathered}\left(\mathrm{Sym}^2 \mathrm{St} \otimes \mathbf{1} + \mathbf{1} \otimes \mathrm{Sym}^2 \mathrm{St}\right) \otimes \mathbf{1} + (\mathrm{St} \otimes \mathbf{1} + \mathbf{1} \otimes \mathrm{St}) \otimes \mathrm{V}_4 \\ + \mathrm{St} \otimes \mathrm{St} \otimes \wedge^* \mathrm{V}_4 + \mathbf{1} \otimes \mathbf{1} \otimes \mathrm{Sym}^2 \mathrm{V}_4,\end{gathered}$$

where V_4 is the standard representation of Sp(2) and $\wedge^* \mathrm{V}_4$ is the 5-dimensional irreducible representation of Sp(2).

4.6.7 $\mathrm{A}_1^{[7^3,1^5]} \times \mathrm{A}_1^{[5,3^7]}$

The centralizer of H in F_4 is trivial.

The restriction of J_0 to H is isomorphic to

$$\mathrm{Sym}^6 \mathrm{St} \otimes \mathrm{Sym}^2 \mathrm{St} + \mathbf{1} \otimes \mathrm{Sym}^4 \mathrm{St},$$

and the restriction of $\mathfrak{f}_4$ to H is isomorphic to

$$\left(\mathrm{Sym}^{10} \mathrm{St} + \mathrm{Sym}^2 \mathrm{St}\right) \otimes \mathbf{1} + \mathbf{1} \otimes \mathrm{Sym}^2 \mathrm{St} + \mathrm{Sym}^6 \mathrm{St} \otimes \mathrm{Sym}^4 \mathrm{St}.$$

4.6.8 $\mathrm{A}_1^{[5,3^7]} \times \left(\mathrm{A}_1^{[3^3,2^6,1^5]} \times \mathrm{A}_1^{[2^6,1^{14}]}\right)/\mu_2^{\Delta}$

The centralizer of H in F_4 is the center of H, which is a cyclic group of order 2.

The restriction of J_0 to H is isomorphic to

$$\mathrm{Sym}^4 \mathrm{St} \otimes \mathbf{1} \otimes \mathbf{1} + \mathrm{Sym}^2 \mathrm{St} \otimes \left(\mathrm{St} \otimes \mathrm{St} + \mathrm{Sym}^2 \mathrm{St} \otimes \mathbf{1}\right),$$

and the restriction of $\mathfrak{f}_4$ to H is isomorphic to

$$\begin{gathered}\mathrm{Sym}^4 \mathrm{St} \otimes \left(\mathrm{St} \otimes \mathrm{St} + \mathrm{Sym}^2 \mathrm{St} \otimes \mathbf{1}\right) + \mathrm{Sym}^2 \mathrm{St} \otimes \mathbf{1} \otimes \mathbf{1} \\ + \mathbf{1} \otimes \left(\mathrm{Sym}^2 \mathrm{St} \otimes \mathbf{1} + \mathbf{1} \otimes \mathrm{Sym}^2 \mathrm{St} + \mathrm{Sym}^3 \mathrm{St} \otimes \mathrm{St}\right).\end{gathered}$$

4.6.9 $\left(\mathrm{A}_1^{[5,4^4,1^5]} \times \mathrm{A}_1^{[2^6,1^{14}]} \times \mathrm{A}_1^{[2^6,1^{14}]}\right)/\mu_2^{\Delta}$

The centralizer of H in F_4 is the center of H, which is isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.
The restriction of J_0 to H is isomorphic to

$$\mathbf{1} + \mathbf{1} \otimes \mathrm{St} \otimes \mathrm{St} + \mathrm{Sym}^3 \mathrm{St} \otimes (\mathrm{St} \otimes \mathbf{1} + \mathbf{1} \otimes \mathrm{St}) + \mathrm{Sym}^4 \mathrm{St} \otimes \mathbf{1} \otimes \mathbf{1},$$

and the restriction of $\mathfrak{f}_4$ to H is isomorphic to

$$\mathbf{1} \otimes \left(\mathrm{Sym}^2 \mathrm{St} \otimes \mathbf{1} + \mathbf{1} \otimes \mathrm{Sym}^2 \mathrm{St}\right) + \mathrm{Sym}^2 \mathrm{St} \otimes \mathbf{1} \otimes \mathbf{1} + \mathrm{Sym}^3 \mathrm{St} \otimes (\mathrm{St} \otimes \mathbf{1} + \mathbf{1} \otimes \mathrm{St}) \\ + \mathrm{Sym}^4 \mathrm{St} \otimes \mathrm{St} \otimes \mathrm{St} + \mathrm{Sym}^6 \mathrm{St} \otimes \mathbf{1} \otimes \mathbf{1}.$$

4.6.10 $\left(\mathrm{A}_1^{[9,6^2,5]} \times \mathrm{A}_1^{[2^6,1^{14}]}\right)/\mu_2^{\Delta}$

The centralizer of H in F_4 is the center of H, which is a cyclic group of order 2.
The restriction of J_0 to H is isomorphic to

$$\mathrm{Sym}^5 \mathrm{St} \otimes \mathrm{St} + \left(\mathrm{Sym}^8 \mathrm{St} + \mathrm{Sym}^4 \mathrm{St}\right) \otimes \mathbf{1},$$

and the restriction of $\mathfrak{f}_4$ to H is isomorphic to

$$\mathbf{1} \otimes \mathrm{Sym}^2 \mathrm{St} + \left(\mathrm{Sym}^9 \mathrm{St} + \mathrm{Sym}^3 \mathrm{St}\right) \otimes \mathrm{St} + \left(\mathrm{Sym}^{10} \mathrm{St} + \mathrm{Sym}^6 \mathrm{St} + \mathrm{Sym}^2 \mathrm{St}\right) \otimes \mathbf{1}.$$

4.6.11 $\left(\mathrm{A}_1^{[5,4^2,3^3,2^2]} \times \mathrm{A}_1^{[3^3,2^6,1^5]}\right)/\mu_2^{\Delta}$

The centralizer of H in F_4 is the center of H, which is a cyclic group of order 2.
The restriction of J_0 to H is isomorphic to

$$\mathrm{Sym}^4 \mathrm{St} \otimes \mathbf{1} + \left(\mathrm{Sym}^3 \mathrm{St} + \mathrm{St}\right) \otimes \mathrm{St} + \mathrm{Sym}^2 \mathrm{St} \otimes \mathrm{Sym}^2 \mathrm{St},$$

and the restriction of $\mathfrak{f}_4$ to H is isomorphic to

$$\mathrm{St} \otimes \mathrm{Sym}^3 \mathrm{St} + \left(\mathrm{Sym}^4 \mathrm{St} + \mathbf{1}\right) \otimes \mathrm{Sym}^2 \mathrm{St} + \left(\mathrm{Sym}^5 \mathrm{St} + \mathrm{Sym}^3 \mathrm{St}\right) \otimes \mathrm{St} + \left(\mathrm{Sym}^2 \mathrm{St}\right)^{\oplus 2} \otimes \mathbf{1}.$$

4.6.12 $\left(\mathrm{A}_1^{[4^2,3^3,2^4,1]} \times \mathrm{A}_1^{[4^2,3^3,2^4,1]}\right)/\mu_2^{\Delta}$

The centralizer of H in F_4 is the center of H, which is a cyclic group of order 2.
The restriction of J_0 to H is isomorphic to

$$\mathbf{1} + \mathrm{Sym}^3 \mathrm{St} \otimes \mathrm{St} + \mathrm{Sym}^2 \mathrm{St} \otimes \mathrm{Sym}^2 \mathrm{St} + \mathrm{St} \otimes \mathrm{Sym}^3 \mathrm{St},$$

and the restriction of $\mathfrak{f}_4$ to H is isomorphic to

$$\left(\mathrm{Sym}^4 \mathrm{St} + \mathbf{1}\right) \otimes \mathrm{Sym}^2 \mathrm{St} + \mathrm{Sym}^2 \mathrm{St} \otimes \left(\mathrm{Sym}^4 \mathrm{St} + \mathbf{1}\right) + \mathrm{Sym}^3 \mathrm{St} \otimes \mathrm{St} + \mathrm{St} \otimes \mathrm{Sym}^3 \mathrm{St}.$$

4.6.13 $\prod_{i=1}^4 \mathrm{A}_1^{[2^6,1^{14}]}/\mu_2^\Delta$

The centralizer of H in F_4 is the center of H, which is isomorphic to $\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$. The restriction of J_0 to H is isomorphic to

$$\mathbf{1}^{\oplus 2}+\sum_{\mathrm{Sym}}\mathrm{St}\otimes\mathrm{St}\otimes\mathbf{1}\otimes\mathbf{1},$$

where the second term stands for the direct sum of tensor products of standard representations at every two copies of $\mathrm{A}_1^{[2^6,1^{14}]}$ in H. The restriction of $\mathfrak{f}_4$ to H is isomorphic to

$$\sum_{\mathrm{Sym}}\mathrm{Sym}^2\,\mathrm{St}\otimes\mathbf{1}\otimes\mathbf{1}\otimes\mathbf{1}+\sum_{\mathrm{Sym}}\mathrm{St}\otimes\mathrm{St}\otimes\mathbf{1}\otimes\mathbf{1}+\mathrm{St}\otimes\mathrm{St}\otimes\mathrm{St}\otimes\mathrm{St}.$$

5 Arthur's conjectures on automorphic representations

In this section, we are going to review the theory of automorphic representations and Arthur's conjectures on discrete automorphic representations. For our purposes, it is enough to restrict to the special case of level 1 algebraic automorphic forms of a reductive group G over $\mathbb{Q}$ admitting a reductive $\mathbb{Z}$-model, as in [CR15; CL19]. We mainly follow these two references.

5.1 A brief review of automorphic representations

In this subsection we give a quick review on automorphic representations, following [CL19, §4.3]. Let G be a connected reductive group over $\mathbb{Q}$ with a reductive $\mathbb{Z}$-model $(\mathscr{G},\mathrm{id})$, and A_G be the maximal $\mathbb{Q}$-split torus of the center $\mathrm{Z}(G)$ of G. Denote by $G(\mathbb{A})^1$ the quotient of $G(\mathbb{A})$ by the neutral component of $A_G(\mathbb{R})$, and consider the adelic quotient

$$[G]:=G(\mathbb{Q})\backslash G(\mathbb{A})^1=G(\mathbb{Q})A_G(\mathbb{R})^\circ\backslash G(\mathbb{A}).$$

We have a left $G(\mathbb{Q})$-invariant right Haar measure μ on $G(\mathbb{A})$ by [Wei53, §II.9], and the volume of $[G]$ is finite with respect to this measure. The topological group $G(\mathbb{A})$ acts on the space $\mathcal{L}(G):=\mathrm{L}^2([G])$ of square-integrable functions on $[G]$ by right translations. Equipped with the *Petersson inner product* defined as

$$\langle f,f'\rangle:=\int\overline{f}f'd\mu,$$

the space $\mathcal{L}(G)$ becomes a unitary representation of $G(\mathbb{A})$. We denote the closure of the sum of all closed and topologically irreducible subrepresentations of $\mathcal{L}(G)$ by $\mathcal{L}_{\mathrm{disc}}(G)$.

Denote by $\Pi(G)$ the set of equivalence classes of irreducible unitary complex representations π of $G(\mathbb{A})$ such that $\pi=\pi_\infty\otimes\pi_f$, where π_∞ is an irreducible unitary representation of $G(\mathbb{R})$, and π_f is a smooth irreducible representation of $G(\mathbb{A}_f)$ satisfying $\pi_f^{\mathscr{G}(\widehat{\mathbb{Z}})}\neq 0$. We have the following decomposition:

$$\mathcal{L}_{\mathrm{disc}}(G)^{\mathscr{G}(\widehat{\mathbb{Z}})}=\overline{\bigoplus_{\pi\in\Pi(G)}}\mathrm{m}(\pi)\,\pi^{\mathscr{G}(\widehat{\mathbb{Z}})}=\overline{\bigoplus_{\pi\in\Pi(G)}}\mathrm{m}(\pi)\,\pi_\infty\otimes\pi_f^{\mathscr{G}(\widehat{\mathbb{Z}})},\tag{5.1}$$

where the integers $\mathrm{m}(\pi) \geq 0$ are finite due to a fundamental result of Harish-Chandra [Cha68, §I.2, Theorem 1]. We call the integer $\mathrm{m}(\pi)$ the *multiplicity* of π in $\mathcal{L}_{\mathrm{disc}}(G)$.

Now we give the definition of level one discrete automorphic representations, and refer to [BJ79, §4] for the general definition of automorphic representations.

Definition 5.1.1. A *level one discrete automorphic representation* is a representation π of $G(\mathbb{A})$ in $\Pi(G)$ such that its multiplicity $\mathrm{m}(\pi)$ in (5.1) is nonzero. We denote the subset of $\Pi(G)$ consisting of level one discrete automorphic representations by $\Pi_{\mathrm{disc}}(G)$.

Notation 5.1.2. Since in this paper we only deal with level one automorphic representations, so we will always omit "level one" from now on.

Definition 5.1.3. A square-integrable Borel function $f : [G] \to \mathbb{C}$ is a *cusp form* if for the unipotent radical U of each proper parabolic subgroup of G, we have

$$\int_{U(\mathbb{Q})\backslash U(\mathbb{A})} f(ug)du = 0$$

for almost all $g \in G(\mathbb{A})$. We denote the subspace of $\mathcal{L}(G)$ consisting of the classes of cusp forms by $\mathcal{L}_{\mathrm{cusp}}(G)$. A discrete automorphic representation is *cuspidal* if it is a subrepresentation of $\mathcal{L}_{\mathrm{cusp}}(G)$, and we denote by $\Pi_{\mathrm{cusp}}(G)$ the subset of $\Pi(G)$ consisting of cuspidal representations.

Remark 5.1.4. A result of Gelfand, Graev and Piatetski-Shapiro [GGP69] asserts that

$$\mathcal{L}_{\mathrm{cusp}}(G) \subset \mathcal{L}_{\mathrm{disc}}(G) \text{ and } \Pi_{\mathrm{cusp}}(G) \subset \Pi_{\mathrm{disc}}(G).$$

When $G(\mathbb{R})$ is compact, every automorphic representation of G is discrete by the Peter-Weyl theorem.

Denote by $\mathrm{H}(G) = \bigotimes_p \mathrm{H}_p(G)$ the *spherical Hecke algebra* of the pair $\left(G(\mathbb{A}_f), \mathscr{G}(\widehat{\mathbb{Z}})\right)$. For any representation $\pi = \pi_\infty \otimes \pi_f \in \Pi(G)$, the space $\pi_f^{\mathscr{G}(\widehat{\mathbb{Z}})}$ is an irreducible representation of the spherical Hecke algebra $\mathrm{H}(G)$. Since $\mathrm{H}(G)$ is commutative [Gro98, Proposition 2.10], the dimension of $\pi_f^{\mathscr{G}(\widehat{\mathbb{Z}})}$ is 1. Hence the $\mathscr{G}(\widehat{\mathbb{Z}})$-invariant space of the π-isotypic subspace $\mathcal{L}_{\mathrm{disc}}(G)_\pi$ of $\mathcal{L}_{\mathrm{disc}}(G)$, as a $G(\mathbb{R})$-representation, is the direct sum of $\mathrm{m}(\pi)$ copies of π_∞. This implies the following result:

Lemma 5.1.5. *Let V be an irreducible unitary representation of the Lie group $G(\mathbb{R})$, and $\mathcal{A}_V(G)$ the space of $G(\mathbb{R})$-equivariant linear maps from V to $\mathcal{L}_{\mathrm{disc}}(G)^{\mathscr{G}(\widehat{\mathbb{Z}})}$. Then we have the following equality:*

$$\dim \mathcal{A}_V(G) = \sum_{\pi \in \Pi(G),\, \pi_\infty \simeq V} \mathrm{m}(\pi). \tag{5.2}$$

Remark 5.1.6. The space $\mathcal{A}_V(G) = \mathrm{Hom}_{G(\mathbb{R})}(V, \mathcal{L}_{\mathrm{disc}}(G)^{\mathscr{G}(\widehat{\mathbb{Z}})})$ can be viewed as the multiplicity space of V in (5.1).

5.1.1 Automorphic representations for $\mathbf{F}_4$

When the reductive group G has compact real points, due to [Gro99a] we can describe the multiplicity space $\mathcal{A}_V(G)$ of V in $\mathcal{L}_{\mathrm{disc}}(G)^{\mathscr{G}(\widehat{\mathbb{Z}})}$ in a more computable manner, which is explained in [CL19, §4.4.1]. Applying [CL19, Lemma 4.4.2] to $\mathbf{F}_4$ and using the fact that every irreducible representation of F_4 is self-dual, we get:

Proposition 5.1.7. *Let (ρ, V) be an irreducible representation of $\mathrm{F}_4 = \mathbf{F}_4(\mathbb{R})$. The vector space $\mathcal{A}_V(\mathbf{F}_4)$ is canonically isomorphic to the following space:*

$$\mathrm{M}_V(\mathbf{F}_4) := \left\{ f : \mathbf{F}_4(\mathbb{A}_f)/\mathcal{F}_{4,\mathrm{I}}(\widehat{\mathbb{Z}}) \to V \,\middle|\, f(\gamma g) = \rho(\gamma) f(g) \text{ for all } \gamma \in \mathbf{F}_4(\mathbb{Q}), g \in \mathbf{F}_4(\mathbb{A}_f) \right\}.$$

We choose a set of representatives $\{1, g_{\mathrm{E}}\}$ of $\mathbf{F}_4(\mathbb{Q})\backslash\mathbf{F}_4(\mathbb{A}_f)/\mathcal{F}_{4,\mathrm{I}}(\widehat{\mathbb{Z}})$ corresponding to the two reductive $\mathbb{Z}$-models $(\mathcal{F}_{4,\mathrm{I}}, \mathrm{id})$ and $(\mathcal{F}_{4,\mathrm{E}}, \iota)$ of $\mathbf{F}_4$ in Proposition 2.3.5. By [CL19, Equation (4.4.1)] the evaluation map $f \mapsto (f(1), f(g_{\mathrm{E}}))$ induces a bijection:

$$\mathrm{M}_V(\mathbf{F}_4) \simeq V^{\mathcal{F}_{4,\mathrm{I}}(\mathbb{Z})} \oplus V^{\mathcal{F}_{4,\mathrm{E}}(\mathbb{Z})}.$$

Combining the results in this section with Theorem 3.6.1, we have the following computational result:

Corollary 5.1.8. *For any dominant weight λ of F_4, we have an explicit formula for $\dim \mathcal{A}_{\mathrm{V}_\lambda}(\mathbf{F}_4)$, where V_λ is the irreducible representation of $\mathrm{F}_4 = \mathbf{F}_4(\mathbb{R})$ with highest weight λ. For $\lambda = (\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ with $2\lambda_1 + 3\lambda_2 + 2\lambda_3 + \lambda_4 \leq 13$, the dimension $\dim \mathcal{A}_{\mathrm{V}_\lambda}(\mathbf{F}_4)$ equals the $d(\lambda)$ in Table 6.*

5.2 Local parametrization of $\Pi(G)$

Let G be a connected reductive group over $\mathbb{Q}$ with a fixed reductive $\mathbb{Z}$-model $(\mathscr{G}, \mathrm{id})$. Let $\widehat{G}$ be its complex Langlands dual group, i.e. the root datum of $\widehat{G}$ is the dual root datum of G. A representation $\pi \in \Pi(G)$ can be decomposed as $\pi = \pi_\infty \otimes \left(\bigotimes_p \pi_p\right)$, where π_p is a *spherical* irreducible smooth representation of $G(\mathbb{Q}_p)$ for each p, i.e. $\pi_p^{\mathscr{G}(\mathbb{Z}_p)} \neq 0$, and π_∞ is an irreducible unitary representation of the Lie group $G(\mathbb{R})$.

In this subsection, we will recall the parametrizations for spherical irreducible smooth representations of $G(\mathbb{Q}_p)$ and for irreducible unitary representations of $G(\mathbb{R})$. Our main reference is [CL19, §6.2, §6.3].

5.2.1 Satake parameter

For each prime number p, a spherical irreducible smooth representation π of $G(\mathbb{Q}_p)$ is determined by the action of the spherical Hecke algebra $\mathrm{H}_p(G)$ for the pair $(G(\mathbb{Q}_p), \mathscr{G}(\mathbb{Z}_p))$ on the subspace of invariants $\pi^{\mathscr{G}(\mathbb{Z}_p)}$. Since $\dim \pi^{\mathscr{G}(\mathbb{Z}_p)} = 1$, the equivalence class of π is determined uniquely by the ring homomorphism $\mathrm{H}_p(G) \to \mathbb{C}$ given by the $\mathrm{H}_p(G)$-action on $\pi^{\mathscr{G}(\mathbb{Z}_p)}$.

By [CL19, Scholium 6.2.2], the *Satake isomorphism* gives a canonical bijection between the set of ring homomorphisms $\mathrm{H}_p(G) \to \mathbb{C}$ and the set $\widehat{G}(\mathbb{C})_{\mathrm{ss}}$ of semisimple conjugacy classes in $\widehat{G}(\mathbb{C})$. This induces a bijection $\pi \mapsto \mathrm{c}_p(\pi)$ between the set of equivalence classes of spherical irreducible smooth representations of $G(\mathbb{Q}_p)$ and the set $\widehat{G}(\mathbb{C})_{\mathrm{ss}}$. The conjugacy class $\mathrm{c}_p(\pi)$ is called the *Satake parameter* of π_p.

5.2.2 Infinitesimal character

Let $\mathfrak{g}$ be the Lie algebra of $G(\mathbb{C})$, and $\widehat{\mathfrak{g}}$ the Lie algebra of $\widehat{G}(\mathbb{C})$. We fix a Cartan subalgebra $\mathfrak{t}$ of $\mathfrak{g}$ and a Borel subalgebra $\mathfrak{b} \subset \mathfrak{g}$ containing $\mathfrak{t}$, and denote the Weyl group of $\mathfrak{g}$ with respect to $\mathfrak{t}$ by W.

As explained in [CL19, §6.3.4], we can associate a character $\mathrm{Z}(\mathrm{U}(\mathfrak{g})) \to \mathbb{C}$ to an irreducible unitary representation (π, V) of $G(\mathbb{R})$, where $\mathrm{Z}(\mathrm{U}(\mathfrak{g}))$ is the center of the universal enveloping algebra of $\mathfrak{g}$. By [CL19, Scholium 6.3.2 and Equation (6.3.1)], the *Harish-Chandra isomorphism* induces the following canonical bijections:

$$\mathrm{Hom}_{\mathbb{C}\text{-alg}}(\mathrm{Z}(\mathrm{U}(\mathfrak{g})), \mathbb{C}) \simeq \widehat{\mathfrak{g}}_{\mathrm{ss}} \simeq \left(X^*(\mathfrak{t}) \otimes_{\mathbb{Z}} \mathbb{C}\right)/W, \tag{5.3}$$

where $\widehat{\mathfrak{g}}_{\mathrm{ss}}$ is the set of semisimple conjugacy classes in $\widehat{\mathfrak{g}}$. Hence we associate to (π, V) a semisimple conjugacy class $\mathrm{c}_\infty(\pi) \in \widehat{\mathfrak{g}}_{\mathrm{ss}}$, called the *infinitesimal character* of π.

As proved by Harish-Chandra [Kna86, Corollary 10.37], up to isomorphism there are only a finite number of irreducible unitary representations of $G(\mathbb{R})$ with a given infinitesimal character. When $G(\mathbb{R})$ is compact, the situation is much simpler due to the following result:

Proposition 5.2.1. *[Dix77, §7.4.6] Let $G(\mathbb{R})$ be a compact group, and $\rho \in X^*(\mathfrak{t}) \otimes \mathbb{C}$ the half-sum of positive roots with respect to $(\mathfrak{g}, \mathfrak{b}, \mathfrak{t})$. For a dominant weight λ of $G(\mathbb{R})$, the infinitesimal character of the highest weight representation V_λ of $G(\mathbb{R})$ is $\lambda + \rho$, viewed as an element in $\widehat{\mathfrak{g}}_{\mathrm{ss}}$ via (5.3). In particular, the infinitesimal character $\lambda + \rho$ determines V_λ uniquely.*

5.2.3 Langlands parametrization

Now we recall Langlands parametrization of $\Pi(G)$, following [CL19, §6.4.2].

Definition 5.2.2. Let H be a connected reductive $\mathbb{C}$-group with complex Lie algebra $\mathfrak{h}$. We denote by $H(\mathbb{C})_{\mathrm{ss}}$ (resp. $\mathfrak{h}_{\mathrm{ss}}$) the set of $H(\mathbb{C})$-conjugacy classes of semisimple elements of $H(\mathbb{C})$ (resp. $\mathfrak{h}$). We denote by $\mathcal{X}(H)$ the set of families $(\mathrm{c}_\infty, \mathrm{c}_2, \mathrm{c}_3, \mathrm{c}_5, \ldots)$, where $\mathrm{c}_\infty \in \mathfrak{h}_{\mathrm{ss}}$ and $\mathrm{c}_p \in H(\mathbb{C})_{\mathrm{ss}}$ for all primes p.

By results in §5.2.1 and §5.2.2, we associate to a representation $\pi = \pi_\infty \otimes \left(\bigotimes_p \pi_p\right) \in \Pi(G)$ a conjugacy class $\mathrm{c}_p(\pi) := \mathrm{c}_p(\pi_p)$ in $\widehat{G}(\mathbb{C})_{\mathrm{ss}}$ for each p, and a conjugacy class $\mathrm{c}_\infty(\pi) := \mathrm{c}_\infty(\pi_\infty)$ in $\widehat{\mathfrak{g}}_{\mathrm{ss}}$. Hence we have a canonical map $\Pi(G) \to \mathcal{X}(\widehat{G})$ defined as

$$\pi = \pi_\infty \otimes \left(\bigotimes_p \pi_p\right) \mapsto \mathrm{c}(\pi) = (\mathrm{c}_\infty(\pi), \mathrm{c}_2(\pi), \mathrm{c}_3(\pi), \cdots) \in \mathcal{X}(\widehat{G}).$$

The family of conjugacy classes $\mathrm{c}(\pi)$ determines π_f and the infinitesimal character of π_∞, and the map c has finite fibers. When $G(\mathbb{R})$ is compact, the fiber of c is either empty or a singleton.

Definition 5.2.3. Let G be a semisimple $\mathbb{Q}$-group admitting a reductive $\mathbb{Z}$-model, and $r : \widehat{G} \to \mathrm{SL}_n$ an algebraic representation of its dual group, which induces a map $\mathcal{X}(\widehat{G}) \to \mathcal{X}(\mathrm{SL}_n)$. For any $\pi \in \Pi(\mathscr{G})$, we define the following family of conjugacy classes:

$$\psi(\pi, r) := r\left(\mathrm{c}(\pi)\right) \in \mathcal{X}(\mathrm{SL}_n),$$

and refer to it as the *Langlands parameter of the pair* (π, r).

5.3 Global parametrization and the Langlands group

For the global parametrization of level one discrete automorphic representations, now we need to use a *conjectural* group $\mathcal{L}_{\mathbb{Z}}$, the so-called *Langlands group of* $\mathbb{Z}$, to formulate the global Arthur-Langlands conjecture. In Arthur's work [Art89], he uses another group $\mathcal{L}_{\mathbb{Q}}$. However, since we only consider level one discrete automorphic representations in this paper, it is more convenient to use the group $\mathcal{L}_{\mathbb{Z}}$ that we are going to recall, following [CR15, Appendix B; CL19, Preface].

We assume that $\mathcal{L}_{\mathbb{Z}}$ is a compact Hausdorff topological group equipped with

- A conjugacy class Frob_p in $\mathcal{L}_{\mathbb{Z}}$, for each prime p,
- A conjugacy class of continuous homomorphisms $\mathrm{h} : \mathrm{W}_{\mathbb{R}} \to \mathcal{L}_{\mathbb{Z}}$, called the *Hodge morphism*. Here $\mathrm{W}_{\mathbb{R}}$ is the *Weil group of* $\mathbb{R}$, which is a non-split extension of $\mathrm{Gal}(\mathbb{C}/\mathbb{R}) = \{1, j\}$ by $\mathrm{W}_{\mathbb{C}} = \mathbb{C}^{\times}$, for the natural action of $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$ on $\mathbb{C}^{\times}$. It is generated by its open subgroup $\mathbb{C}^{\times}$ together with an element j, with relations $j^2 = -1$ and $jzj^{-1} = \overline{z}$ for every $z \in \mathbb{C}^{\times}$.

This group $\mathcal{L}_{\mathbb{Z}}$ satisfies three axioms that we will introduce one by one.

Axiom 1. *(*Cebotarev property*) The union of conjugacy classes* Frob_p *is dense in* $\mathcal{L}_{\mathbb{Z}}$*.*

Remark 5.3.1. In [CR15, Appendix B], the axiom they use is the *general Sato-Tate conjecture*: the conjugacy classes Frob_p are equidistributed in the compact group $\mathcal{L}_{\mathbb{Z}}$ equipped with its Haar measure of mass 1. This is a universal form of the Sato-Tate conjecture for automorphic representations and it implies the Cebotarev property, but Axiom 1 is enough for us in this article.

This axiom tells us for two homomorphisms ψ, ψ' from $\mathcal{L}_{\mathbb{Z}}$ to some topological group H, if $\psi(\mathrm{Frob}_p)$ and $\psi'(\mathrm{Frob}_p)$ are conjugate in H for each prime p, then ψ and ψ' are element-conjugate. An important type of homomorphisms involving $\mathcal{L}_{\mathbb{Z}}$ is:

Definition 5.3.2. Let G be a reductive $\mathbb{Q}$-group admitting a reductive $\mathbb{Z}$-model. A *discrete global Arthur parameter (of level one)* of G is a $\widehat{G}(\mathbb{C})$-conjugacy class of continuous group homomorphisms

$$\psi : \mathcal{L}_{\mathbb{Z}} \times \mathrm{SL}_2(\mathbb{C}) \to \widehat{G}(\mathbb{C})$$

such that $\psi|_{\mathrm{SL}_2(\mathbb{C})}$ is algebraic and the centralizer C_{ψ} of $\mathrm{Im}(\psi)$ in $\widehat{G}(\mathbb{C})$ is finite modulo the center of $\widehat{G}(\mathbb{C})$. We call C_{ψ} the *(global) component group* of ψ, and denote the set of discrete global Arthur parameters of G by $\Psi_{\mathrm{disc}}(G)$.

Remark 5.3.3. The condition on C_ψ in Definition 5.3.2 implies that a discrete global Arthur parameter for $G = \mathrm{GL}_n$ is an equivalence class of n-dimensional irreducible representations of $\mathcal{L}_\mathbb{Z} \times \mathrm{SL}_2(\mathbb{C})$.

In parallel with Langlands parametrization in §5.2.3, we can also associate to any $\psi \in \Psi_{\mathrm{disc}}(G)$ a collection of conjugacy classes $\mathrm{c}(\psi) = (\mathrm{c}_\infty(\psi), \mathrm{c}_2(\psi), \mathrm{c}_3(\psi), \cdots) \in \mathcal{X}(\widehat{G})$. For each prime p, the conjugacy class $\mathrm{c}_p(\psi)$ is defined by:

$$\mathrm{c}_p(\psi) := \psi(\mathrm{Frob}_p, e_p),\ e_p = \begin{pmatrix} p^{-1/2} & 0 \\ 0 & p^{1/2} \end{pmatrix} \in \mathrm{SL}_2(\mathbb{C}).$$

The infinitesimal character $\mathrm{c}_\infty(\psi)$ of ψ is defined to be the infinitesimal character of the archimedean Arthur parameter $\psi \circ (\mathrm{h} \times \mathrm{id}) : \mathrm{W}_\mathbb{R} \times \mathrm{SL}_2(\mathbb{C}) \to \widehat{G}(\mathbb{C})$, which is explained in [CR15, §A.2].

The following axiom connects the collection of conjugacy classes attached to a discrete automorphic representation and that attached to a discrete global Arthur parameter.

Axiom 2. *(Arthur-Langlands conjecture for GL_n) For every integer $n \geq 1$, there is a unique bijection*

$$\Pi_{\mathrm{disc}}(\mathrm{GL}_n) \xrightarrow{\sim} \Psi_{\mathrm{disc}}(\mathrm{GL}_n),\ \pi \mapsto \psi_\pi$$

such that $\mathrm{c}(\pi) = \mathrm{c}(\psi_\pi)$ for all discrete automorphic representations π of GL_n. Moreover, the discrete global Arthur parameter ψ_π is trivial on $\mathrm{SL}_2(\mathbb{C})$ if and only if we have $\pi \in \Pi_{\mathrm{cusp}}(\mathrm{GL}_n)$.

Remark 5.3.4. This axiom and the compactness of $\mathcal{L}_\mathbb{Z}$ imply the so-called *generalized Ramanujan conjecture*: for any $\pi \in \Pi_{\mathrm{cusp}}(\mathrm{GL}_n)$ and any prime p, the eigenvalues of $\mathrm{c}_p(\pi)$ all have absolute value 1.

For general reductive groups, we have the following third axiom:

Axiom 3. *Let G be a reductive group admitting a reductive $\mathbb{Z}$-model $(\mathscr{G}, \mathrm{id})$, then there exists a decomposition*

$$\mathcal{L}_{\mathrm{disc}}(G)^{\mathscr{G}(\widehat{\mathbb{Z}})} = \bigoplus^{\perp}_{\psi \in \Psi_{\mathrm{disc}}(G)} \mathcal{A}_\psi(G), \tag{5.4}$$

stable under the actions of $G(\mathbb{R})$ and $\mathrm{H}(G)$, and satisfying the following property: for $\pi \in \Pi(G)$, if $\pi^{\mathscr{G}(\widehat{\mathbb{Z}})}$ appears in $\mathcal{A}_\psi(G)$, then we have $\mathrm{c}(\pi) = \mathrm{c}(\psi)$.

This axiom tells us for any level one discrete automorphic representation $\pi \in \Pi_{\mathrm{disc}}(G)$, there exists a discrete global Arthur parameter ψ of G such that $\mathrm{c}(\psi) = \mathrm{c}(\pi)$. In general, this discrete global Arthur parameter is not unique since two element-conjugate embeddings into $\widehat{G}(\mathbb{C})$ may not be conjugate. Conversely, given a discrete global Arthur parameter ψ of G, there are finitely many (possibly zero) adelic representations $\pi \in \Pi(G)$ satisfying $\mathrm{c}(\pi) = \mathrm{c}(\psi)$, and we denote the subset of $\Pi(G)$ consisting of such representations by $\Pi_\psi(G)$.

In other words, discrete global Arthur parameters are the objects parametrizing discrete automorphic representations, but a natural problem that we need to deal with is that which representations in $\Pi_\psi(G)$ for a given ψ appear in the discrete spectrum $\mathcal{L}(G)_{\mathrm{disc}}$. We will see the (conjectural) answer in §5.6.

Another property about $\mathcal{L}_\mathbb{Z}$ that we will use is that it is connected:

Proposition 5.3.5. *[CL19, Proposition 9.3.4] Suppose that $\mathcal{L}_\mathbb{Z}$ is a compact topological group satisfying the axioms above, then it is connected.*

5.3.1 Sato-Tate group

For a discrete global Arthur parameter $\psi \in \Psi_{\text{disc}}(G)$, we pick a representative $\mathcal{L}_{\mathbb{Z}} \times \mathrm{SL}_2(\mathbb{C}) \to \widehat{G}(\mathbb{C})$ and consider its restriction to a maximal compact subgroup:

$$\psi_{\text{c}} : \mathcal{L}_{\mathbb{Z}} \times \mathrm{SU}(2) \to \widehat{G}(\mathbb{C}).$$

The image of this morphism is contained in some maximal compact subgroup of $\widehat{G}(\mathbb{C})$. Fix a maximal connected compact subgroup K of $\widehat{G}(\mathbb{C})$, and without loss of generality we assume that ψ_{c} is a morphism from $\mathcal{L}_{\mathbb{Z}} \times \mathrm{SU}(2) \to K$.

Definition 5.3.6. For any $\psi \in \Psi_{\text{disc}}(G)$, we define $\mathrm{H}(\psi)$ to be the K-conjugacy class of the image of its associated morphism $\mathcal{L}_{\mathbb{Z}} \times \mathrm{SU}(2) \to K$. For any $\pi \in \Pi_{\text{disc}}(G)$, if there exists a unique global Arthur parameter $\psi_\pi \in \Psi_{\text{disc}}(G)$ such that $\mathrm{c}(\pi) = \mathrm{c}(\psi_\pi)$, we define $\mathrm{H}(\pi)$ to be $\mathrm{H}(\psi_\pi)$.

Remark 5.3.7. Since maximal connected compact subgroups of $\mathrm{SL}_2(\mathbb{C})$ are unique up to conjugacy, the $\widehat{G}(\mathbb{C})$-conjugacy class of the image of $\mathcal{L}_{\mathbb{Z}} \times \mathrm{SU}(2) \to K$ is well-defined. Combining with [FHS16, Lemma 2.4], the K-conjugacy class $\mathrm{H}(\psi)$ is well-defined.

Remark 5.3.8. The conjugacy class $\mathrm{H}(\psi)$, or $\mathrm{H}(\pi)$, of subgroups of K is called the *"Sato-Tate group"* in the introduction §1, although it coincides with the usual Sato-Tate group (see [CR15, Proposition-Definition B.1]) if and only if the restriction of ψ to $\mathrm{SL}_2(\mathbb{C})$ is trivial.

A cuspidal automorphic representation π of PGL_n can be viewed as an element of $\Pi_{\text{cusp}}(\mathrm{GL}_n)$ with trivial central character, and the global Arthur parameter ψ_π associated to π via Axiom 2 takes value in $\mathrm{SL}_n(\mathbb{C}) = \widehat{\mathrm{PGL}_n}(\mathbb{C})$. In this case, the global Arthur parameter ψ_π is trivial on $\mathrm{SL}_2(\mathbb{C})$, and the conjugacy class $\mathrm{H}(\pi)$ of subgroups of $\mathrm{SU}(n)$ coincides with the usual Sato-Tate group of π.

5.4 Cuspidal representations of GL_n

Arthur's classification of automorphic representations involves self-dual cuspidal representations of $\mathrm{GL}_n, n \geq 1$. Moreover, these representations of GL_n are trivial on the center of GL_n when they have level one, thus we can replace GL_n by PGL_n. In this subsection we will say more about this class of automorphic representations.

Definition 5.4.1. A representation $\pi \in \Pi_{\text{cusp}}(\mathrm{PGL}_n)$ is *self-dual* if it is isomorphic to its dual representation $\pi^\vee$, and we denote the subset of $\Pi_{\text{cusp}}(\mathrm{PGL}_n)$ consisting of self-dual representations by $\Pi_{\text{cusp}}^\perp(\mathrm{PGL}_n)$.

Remark 5.4.2. By the multiplicity one theorem of Jacquet-Shalika, this self-dual condition is equivalent to that $\mathrm{c}_p(\pi) = \mathrm{c}_p(\pi)^{-1}$ for each prime p and $\mathrm{c}_\infty(\pi) = -\mathrm{c}_\infty(\pi)$.

For a representation $\pi \in \Pi_{\text{cusp}}(\mathrm{PGL}_n)$, its infinitesimal character $\mathrm{c}_\infty(\pi)$ is a conjugacy class in $\mathfrak{sl}_n$. Denote by $\mathrm{Weights}(\pi)$ the multiset of eigenvalues of $\mathrm{c}_\infty(\pi)$.

Definition 5.4.3. A cuspidal automorphic representation π of PGL_n is

- *algebraic* [9] if $\mathrm{Weights}(\pi) \subset \frac{1}{2}\mathbb{Z}$ and for any $w, w' \in \mathrm{Weights}(\pi)$ we have $w - w' \in \mathbb{Z}$.
- *regular* if $|\mathrm{Weights}(\pi)| = n$.

We denote by $\Pi^{\perp}_{\mathrm{alg}}(\mathrm{PGL}_n)$ the subset of $\Pi^{\perp}_{\mathrm{cusp}}(\mathrm{PGL}_n)$ consisting of algebraic representations, and by $\Pi^{\perp}_{\mathrm{alg,reg}}(\mathrm{PGL}_n)$ the subset consisting of algebraic regular representations.

For an algebraic self-dual cuspidal representation π of PGL_n, let $k_1 \geq k_2 \geq \cdots \geq k_n$ be the weights of π (counted with multiplicity). Since π is self-dual, we have $k_i = -k_{n+1-i}$ for $i = 1, 2, \ldots, n$. Following [CR15, §1.5], we call the integers

$$w_i = 2k_i,\ i = 1, 2, \ldots, [n/2]$$

the *Hodge weights* of π and call the maximal Hodge weight $\mathrm{w}(\pi) := w_1$ the *motivic weight* of π.

5.4.1 Arthur's orthogonal-symplectic alternative

We can divide the set self-dual cuspidal representations of PGL_n into two parts, by Arthur's *symplectic-orthogonal alternative*. Our reference is [CL19, §8.3.1].

The classical groups over $\mathbb{Z}$ that are Chevalley groups are therefore Sp_{2g} for $g \geq 1$, $\mathrm{SO}_{r,r}$ for $r \geq 2$, and $\mathrm{SO}_{r+1,r}$ for $r \geq 1$. For one of these groups G, we denote the standard representation of $\widehat{G}(\mathbb{C})$ by $\mathrm{St} : \widehat{G}(\mathbb{C}) \hookrightarrow \mathrm{SL}_{\mathrm{n}(G)}(\mathbb{C})$. For instance, $\mathrm{n}(\mathrm{Sp}_{2g}) = 2g+1$, $\mathrm{n}(\mathrm{SO}_{r,r}) = 2r$ and $\mathrm{n}(\mathrm{SO}_{r+1,r}) = 2r$. This map St also induces a natural map from $\mathcal{X}(\widehat{G})$ to $\mathcal{X}(\mathrm{SL}_{\mathrm{n}(G)})$. We have the following theorem by Arthur:

Theorem 5.4.4. *[Art13, Theorem 1.4.1] For any $n \geq 1$ and a self-dual cuspidal representation π of PGL_n, there exists a classical Chevalley group G^{π}, unique up to isomorphism, with the following properties:*

(i) We have $\mathrm{n}(G^{\pi}) = n$.
(ii) There exists a representation $\pi' \in \Pi_{\mathrm{disc}}(G^{\pi})$ such that $\psi(\pi', \mathrm{St}) = \mathrm{c}(\pi)$.

Definition 5.4.5. A representation $\pi \in \Pi^{\perp}_{\mathrm{cusp}}(\mathrm{PGL}_n)$ is called *orthogonal* if $\widehat{G^{\pi}}(\mathbb{C}) \simeq \mathrm{SO}_n(\mathbb{C})$ and *symplectic* otherwise. We denote the subset of $\Pi^{\perp}_{\mathrm{cusp}}(\mathrm{PGL}_n)$ consisting of orthogonal representations by $\Pi^{\mathrm{o}}_{\mathrm{cusp}}(\mathrm{PGL}_n)$, and the subset consisting of symplectic representations by $\Pi^{\mathrm{s}}_{\mathrm{cusp}}(\mathrm{PGL}_n)$.

For $* = \mathrm{alg}$ or $\mathrm{alg,reg}$, we define $\Pi^{\mathrm{o}}_{*}(\mathrm{PGL}_n) = \Pi^{\mathrm{o}}_{\mathrm{cusp}}(\mathrm{PGL}_n) \cap \Pi^{\perp}_{*}(\mathrm{PGL}_n)$ and $\Pi^{\mathrm{s}}_{*}(\mathrm{PGL}_n) = \Pi^{\mathrm{s}}_{\mathrm{cusp}}(\mathrm{PGL}_n) \cap \Pi^{\perp}_{*}(\mathrm{PGL}_n)$. We define the subset $\Pi^{\mathrm{Sp}_{2n}}_{\mathrm{alg}}(\mathrm{PGL}_{2n}) \subset \Pi^{\mathrm{s}}_{\mathrm{alg,reg}}(\mathrm{PGL}_{2n})$ as:

$$\left\{ \pi \in \Pi^{\mathrm{s}}_{\mathrm{alg,reg}}(\mathrm{PGL}_{2n}) \,\middle|\, \mathrm{Im}(\psi_{\pi}) \simeq \mathrm{Sp}(n) \right\},$$

and similarly define

$$\Pi^{\mathrm{SO}_n}_{\mathrm{alg}}(\mathrm{PGL}_n) = \left\{ \pi \in \Pi^{\mathrm{o}}_{\mathrm{alg,reg}}(\mathrm{PGL}_n) \,\middle|\, \mathrm{Im}(\psi_{\pi}) \simeq \mathrm{SO}(n) \right\}.$$

Example 5.4.6. A representation $\pi \in \Pi_{\mathrm{cusp}}(\mathrm{PGL}_2)$ is necessarily self-dual and symplectic, thus $\Pi_{\mathrm{cusp}}(\mathrm{PGL}_2) = \Pi^{\perp}_{\mathrm{cusp}}(\mathrm{PGL}_2) = \Pi^{\mathrm{s}}_{\mathrm{cusp}}(\mathrm{PGL}_2)$. Moreover, for each positive integer w we have a bijection between the set of level one normalized Hecke eigenforms of weight $w+1$ and the set of $\pi \in \Pi^{\perp}_{\mathrm{alg}}(\mathrm{PGL}_2)$ with Hodge weight w. In particular, level one algebraic cuspidal representations with Hodge weight w exist only when $w \geq 11$.

[9] The term *algebraic* is in the sense of Borel [Bor79, §18.2].

5.4.2 Global ε-factor

An important factor related to a cuspidal representation π is its *global ε-factor* $\varepsilon(\pi)$. We briefly give its definition as follows: for two level one cuspidal representations $\pi \in \Pi_{\mathrm{cusp}}(\mathrm{PGL}_n)$ and $\pi' \in \Pi_{\mathrm{cusp}}(\mathrm{PGL}_{n'})$, Jacquet, Shalika and Piatetski-Shapiro define a factor $\varepsilon(\pi\times\pi')$ when studying the meromorphic continuation and functional equation of the Rankin-Selberg L-function $\mathrm{L}(s,\pi\times\pi')$ [Cog04, §9].

Definition 5.4.7. The *global ε-factor* of $\pi \in \Pi_{\mathrm{cusp}}(\mathrm{PGL}_n)$ is defined as $\varepsilon(\pi) := \varepsilon(\pi \times \mathbf{1})$.

For orthogonal algebraic representations, we have the following result by Arthur:

Theorem 5.4.8. *[Art13, Theorem 1.5.3] If $\pi \in \Pi_{\mathrm{alg}}^{\mathrm{o}}(\mathrm{PGL}_n)$, then $\varepsilon(\pi) = 1$.*

In [CL19, §8.2.21], a method to compute $\varepsilon(\pi)$ for $\pi \in \Pi_{\mathrm{alg}}^{\mathrm{s}}(\mathrm{PGL}_n)$ is explained. To recall that method, we review first the archimedean Local Langlands correspondence [Lan73]. We can associate with each irreducible unitary representation U of $\mathrm{GL}_n(\mathbb{R})$ a unique (up to conjugacy) semisimple representation $\mathrm{L}(U) : \mathrm{W}_{\mathbb{R}} \to \mathrm{GL}_n(\mathbb{C})$. By Clozel's purity lemma [Clo90, Lemma 4.9], for a representation $\pi \in \Pi_{\mathrm{alg}}^{\perp}(\mathrm{PGL}_n)$, the associated representation $\mathrm{L}(\pi_\infty)$ is a direct sum of the following types of irreducible representations:

- the trivial representation $\mathbf{1}$,
- the sign character $\epsilon_{\mathbb{C}/\mathbb{R}} = \eta/|\eta|$,
- and the 2-dimensional induced representation $\mathbf{I}_w := \mathrm{Ind}_{\mathrm{W}_{\mathbb{C}}}^{\mathrm{W}_{\mathbb{R}}}\left(z \mapsto z^{w/2}\overline{z}^{-w/2}\right)$ for some positive integer w, where $z \mapsto z^{w/2}\overline{z}^{-w/2}$ stands for the character $z \mapsto (z/\overline{z})^w$ by an abuse of notation.

There is a unique way to associate a fourth root of unity $\varepsilon(\rho)$ with each $\rho : \mathrm{W}_{\mathbb{R}} \to \mathrm{GL}_n(\mathbb{C})$ of the above forms such that $\varepsilon(\rho \oplus \rho') = \varepsilon(\rho)\varepsilon(\rho')$ and

$$\varepsilon(\mathbf{1}) = 1,\ \varepsilon(\epsilon_{\mathbb{C}/\mathbb{R}}) = i,\ \varepsilon(\mathbf{I}_w) = i^{w+1} \text{ for any integer } w > 0.$$

There is a connection between this factor $\varepsilon\left(\mathrm{L}(\pi_\infty)\right)$ and the global ε-factor of π:

Proposition 5.4.9. *[CL19, Proposition 8.2.22] For $\pi \in \Pi_{\mathrm{alg}}^{\perp}(\mathrm{PGL}_n)$, we have*

$$\varepsilon(\pi) = \varepsilon(\mathrm{L}(\pi_\infty)).$$

As a consequence, we can calculate the global ε-factor of π provided we know the representation $\mathrm{L}(\pi_\infty)$ of $\mathrm{W}_{\mathbb{R}}$ corresponding to π_∞. Actually, one has the following result:

Proposition 5.4.10. *[CL19, Proposition 8.2.13] Let $\pi \in \Pi_{\mathrm{alg}}^{\mathrm{s}}(\mathrm{PGL}_n)$ and $w_1 \geq w_2 \geq \cdots \geq w_{n/2}$ its Hodge weights, then*

$$\mathrm{L}(\pi_\infty) \simeq \mathbf{I}_{w_1} \oplus \mathbf{I}_{w_2} \oplus \cdots \oplus \mathbf{I}_{w_{n/2}}.$$

5.5 Arthur-Langlands conjecture

Assuming the existence of the Langlands group $\mathcal{L}_{\mathbb{Z}}$ described in §5.3. Axiom 3 says that for any reductive group G admitting a reductive $\mathbb{Z}$-model and any discrete automorphic representation π of G, there exists a discrete global Arthur parameter $\psi : \mathcal{L}_{\mathbb{Z}} \times \mathrm{SL}_2(\mathbb{C}) \to \widehat{G}(\mathbb{C})$ such that $\mathrm{c}(\pi) = \mathrm{c}(\psi)$.

Remark 5.5.1. When the group $\widehat{G}(\mathbb{C})$ satisfies the "element-conjugacy implies conjugacy" property as in Proposition 4.1.5, the discrete global Arthur parameter ψ satisfying $\mathrm{c}(\psi) = \mathrm{c}(\pi)$, as a conjugacy class of homomorphisms $\mathcal{L}_{\mathbb{Z}} \times \mathrm{SL}_2(\mathbb{C}) \to \widehat{G}(\mathbb{C})$, is unique.

Let G be semisimple, and fix an irreducible algebraic representation $r : \widehat{G} \to \mathrm{SL}_{n,\mathbb{C}}$. Following [CL19, §6.4.4], we are going to see what the Langlands parameter $\psi(\pi, r)$ defined in Definition 5.2.3 looks like for a discrete automorphic representation π of G.

Composing r with a discrete global Arthur parameter $\psi : \mathcal{L}_{\mathbb{Z}} \times \mathrm{SL}_2(\mathbb{C}) \to \widehat{G}(\mathbb{C})$ corresponding to π, we get an n-dimensional representation $r \circ \psi$ of $\mathcal{L}_{\mathbb{Z}} \times \mathrm{SL}_2(\mathbb{C})$. This representation can be decomposed as

$$\bigoplus_{i=1}^{k} r_i \otimes \mathrm{Sym}^{d_i - 1} \mathrm{St}$$

for some irreducible representations $r_i : \mathcal{L}_{\mathbb{Z}} \to \mathrm{SL}_{n_i}$ and certain integers $d_i \geq 1$, where St denotes the standard 2-dimensional representation of $\mathrm{SL}_2(\mathbb{C})$.

By Arthur-Langlands conjecture for general linear groups, i.e. Axiom 2 in §5.3, every irreducible representation $r_i : \mathcal{L}_{\mathbb{Z}} \to \mathrm{GL}_{n_i}(\mathbb{C})$ corresponds to a unique cuspidal representation π_i of PGL_{n_i}. For $v = p$ or ∞, we have an identity between conjugacy classes:

$$r(\mathrm{c}_v(\pi)) = \bigoplus_{i}^{k} \mathrm{c}_v(\pi_i) \otimes \mathrm{Sym}^{d_i - 1}(e_v).$$

To formulate a global identity, we introduce the following notations:

- Define $e \in \mathcal{X}(\mathrm{SL}_2)$ to be $(e_\infty, e_2, e_3, \cdots)$ and denote $\mathrm{Sym}^{d-1}(e) \in \mathcal{X}(\mathrm{SL}_d)$ by $[d]$.
- Denote by $(c, c') \mapsto c \oplus c'$ the map $\mathcal{X}(\mathrm{SL}_a) \times \mathcal{X}(\mathrm{SL}_b) \to \mathcal{X}(\mathrm{SL}_{a+b})$ induced by the direct sum, and by $(c, c') \mapsto c \otimes c'$ the map $\mathcal{X}(\mathrm{SL}_a) \times \mathcal{X}(\mathrm{SL}_b) \to \mathcal{X}(\mathrm{SL}_{ab})$ induced by the tensor product. We write $c \otimes [d]$ as $c[d]$ for short.
- For $\pi \in \Pi_{\mathrm{cusp}}(\mathrm{PGL}_m)$, the element $\mathrm{c}(\pi) \in \mathcal{X}(\mathrm{SL}_m)$ will simply be denoted by π.

With these notations, we can combine the identities for $r(\mathrm{c}_v(\pi))$ together into one:

$$\psi(\pi, r) = r(\mathrm{c}(\pi)) = \bigoplus_{i=1}^{k} \pi_i[d_i],\ \pi_i \in \Pi_{\mathrm{cusp}}(\mathrm{PGL}_{n_i}).$$

Now we state Arthur-Langlands conjecture for semisimple groups:

Conjecture 5.5.2. (Arthur-Langlands conjecture) *Let G be a semisimple $\mathbb{Q}$-group admitting a reductive $\mathbb{Z}$-model. For any $\pi \in \Pi_{\mathrm{disc}}(G)$ and every algebraic representation $r : \widehat{G} \to \mathrm{SL}_{n,\mathbb{C}}$, there exists a collection of triples $(n_i, \pi_i, d_i)_{i=1,\ldots,k}$ with $d_i, n_i \geq 1$ integers satisfying $n = \sum_i n_i d_i$ and $\pi_i \in \Pi_{\mathrm{cusp}}(\mathrm{PGL}_{n_i})$ such that*

$$\psi(\pi, r) = \pi_1[d_1] \oplus \cdots \oplus \pi_k[d_k].$$

This conjecture was proved by Arthur in [Art13] when G is a split classical group and r is the standard representation of $\widehat{G}$. Moreover, the collection of triples (n_i, π_i, d_i) in the conjecture is necessarily unique up to permutation by a result of Jacquet and Shalika [JS81]:

Proposition 5.5.3. *[CL19, Proposition 6.4.5] Let $k, l \geq 1$ be integers. For $1 \leq i \leq k$ (resp. $1 \leq j \leq l$), consider integers $n_i, d_i \geq 1$ (resp. $n'_j, d'_j \geq 1$) and a representation π_i (resp. π'_j) in $\Pi_{\mathrm{cusp}}(\mathrm{PGL}_{n_i})$ (resp. $\Pi_{\mathrm{cusp}}(\mathrm{PGL}_{n'_j})$). Suppose that we have $n := \sum_i n_i d_i = \sum_j n'_j d'_j$ and*

$$\pi_1[d_1] \oplus \cdots \oplus \pi_k[d_k] = \pi'_1[d'_1] \oplus \cdots \oplus \pi'_l[d'_l].$$

Then $k = l$ and there exists a permutation $\sigma \in \mathrm{S}_k$ such that for every $1 \leq i \leq k$ we have $(n'_i, \pi'_i, d'_i) = (n_{\sigma(i)}, \pi_{\sigma(i)}, d_{\sigma(i)})$.

We call the triple $(k, (n_i, d_i)_{1 \leq i \leq k})$, up to permutations of the (n_i, d_i), the *endoscopic type* of $\psi(\pi, r)$. The parameter is called *stable* if $k = 1$ and *endoscopic* otherwise. It is called *tempered* if $d_i = 1$ for all i and *non-tempered* otherwise.

In Conjecture 5.5.2, cuspidal representations of $\mathrm{PGL}_n, n \geq 1$ are building blocks of Langlands parameters $\psi(\pi, r)$. Furthermore, the following result shows that under some conditions, for example when $G(\mathbb{R})$ is compact, we only need algebraic cuspidal representations:

Proposition 5.5.4. *[CL19, Proposition 8.2.8] Let G be a semisimple $\mathbb{Q}$-group admitting a reductive $\mathbb{Z}$-model, $\pi \in \Pi_{\mathrm{disc}}(G)$ and $r : \widehat{G} \to \mathrm{SL}_{n,\mathbb{C}}$ an n-dimensional algebraic representation of $\widehat{G}$. Suppose that*

(i) $\mathrm{c}_\infty(\pi) \in \widehat{\mathfrak{g}}_{\mathrm{ss}}$ is the infinitesimal character of a finite-dimensional irreducible complex representation of $G_{\mathbb{C}}$,
(ii) and $\psi(\pi, r) = \oplus_{i=1}^k \pi_i[d_i]$ with $\pi_i \in \Pi_{\mathrm{cusp}}(\mathrm{PGL}_{n_i}), i = 1, \ldots, k$.

Then π_i is algebraic for $i = 1, \ldots, k$. Moreover, the class of $\mathrm{w}(\pi_i) + d_i - 1$ in $\mathbb{Z}/2\mathbb{Z}$ depends only on r and not on the integer i or even on π.

5.6 Arthur's multiplicity formula

Arthur gives a *conjectural* formula for the multiplicity of an adelic representation $\pi \in \Pi(G)$ in the discrete spectrum $\mathcal{L}_{\mathrm{disc}}(G)$. In this section, we will state this for a simply-connected anisotropic $\mathbb{Q}$-group G admitting a reductive $\mathbb{Z}$-model, following [Art89, §8].

For a representation $\pi \in \Pi(G)$, there are finitely many discrete global Arthur parameters ψ of G such that $\mathrm{c}(\pi) = \mathrm{c}(\psi)$. According to [Art89], the multiplicity $\mathrm{m}(\pi)$ of π in $\mathcal{L}_{\mathrm{disc}}(G)$ should be the sum of m_ψ over the set of all such ψ, where m_ψ is some integer that we are going to introduce. We note that these ψ all belong to the following subset of $\Psi_{\mathrm{disc}}(G)$:

Definition 5.6.1. We define $\Psi_{\mathrm{AJ}}(G)$ to be the subset of $\Psi_{\mathrm{disc}}(G)$ consisting of $\psi \in \Psi_{\mathrm{disc}}(G)$ satisfying that $\mathrm{c}_\infty(\psi)$ is the infinitesimal character of a finite dimensional irreducible representation of $G_{\mathbb{C}}$.

Remark 5.6.2. The subscript AJ stands for *Adams-Johnson*. This means the archimedean Arthur parameter $\mathrm{W}_{\mathbb{R}} \times \mathrm{SL}_2(\mathbb{C}) \to \widehat{G}(\mathbb{C})$ for $\psi \in \Psi_{\mathrm{disc}}(G)$ is an *Adams-Johnson parameter* in the sense of [CL19, §8.4.14] if and only if $\psi \in \Psi_{\mathrm{AJ}}(G)$. The condition that $\mathrm{c}_\infty(\psi)$ is the infinitesimal character of a finite-dimensional irreducible representation is the condition (AJ1) in [CL19, §8.4.14], and the second condition (AJ2) for Adams-Johnson parameters is automatically satisfied in our case by [Taï17, §4.2.2; NP21, Proposition 6].

Now we let $\psi \in \Psi_{\mathrm{AJ}}(G)$. In Definition 5.3.2, the global component group C_ψ of ψ is defined to be the centralizer of $\mathrm{Im}(\psi)$ in $\widehat{G}(\mathbb{C})$. When G is semisimple, this group is finite since the center of $\widehat{G}$ is finite. Moreover, as explained in [CL19, §8.4.14], C_ψ is an elementary finite abelian 2-group, i.e. a product of finitely many copies of $\mathbb{Z}/2\mathbb{Z}$. For any $\psi \in \Psi_{\mathrm{AJ}}(G)$, Arthur's formula for m_ψ involves two quadratic characters of C_ψ.

5.6.1 The character $\rho_\psi^\vee$

The first character of C_ψ is defined as follows.

By Proposition 5.2.1, the conjugacy class $\mathrm{c}_\infty(\psi)$ for $\psi \in \Psi_{\mathrm{AJ}}(G)$ is regular, viewed as a cocharacter of a maximal torus $\widehat{T}$ of $\widehat{G}$ chosen as in [CL19, §8.4.14]. Hence there is a unique Borel subgroup $\widehat{B} \supset \widehat{T}$ of $\widehat{G}$ with respect to whom the infinitesimal character $\mathrm{c}_\infty(\psi)$ is strictly dominant. Let $\rho_\psi^\vee$ be the half-sum of positive roots with respect to $(\widehat{G}, \widehat{B}, \widehat{T})$. Since G is simply-connected, $\rho_\psi^\vee \in \frac{1}{2}X^*(\widehat{T})$ is a character of $\widehat{T}$. Its restriction to the component group C_ψ is the first character we need, and we denote $\rho^\vee|_{\mathrm{C}_\psi}$ by $\rho_\psi^\vee$ for short.

5.6.2 Arthur's character ε_ψ

A discrete global Arthur parameter $\psi \in \Psi_{\mathrm{AJ}}(G)$ induces a morphism

$$\mathrm{C}_\psi \times \mathcal{L}_\mathbb{Z} \times \mathrm{SL}_2(\mathbb{C}) \to \widehat{G}(\mathbb{C}).$$

Restricting the adjoint representation $\widehat{\mathfrak{g}}$ of $\widehat{G}(\mathbb{C})$ along this morphism, it can be decomposed into a direct sum

$$\widehat{\mathfrak{g}}|_{\mathrm{C}_\psi \times \mathcal{L}_\mathbb{Z} \times \mathrm{SL}_2(\mathbb{C})} = \bigoplus_{i=1}^{l} \chi_i \otimes \pi_i[d_i], \tag{5.5}$$

where χ_i is a quadratic character of C_ψ, and π_i is an n_i-dimensional irreducible representation of $\mathcal{L}_\psi$ which is identified as an element in $\Pi_{\mathrm{cusp}}^\perp(\mathrm{PGL}_{n_i})$. Moreover, since ψ belongs to $\Psi_{\mathrm{AJ}}(G)$, according to Proposition 5.5.4 these cuspidal representations π_i are algebraic.

Definition 5.6.3. [Art89, Equation 8.4] Let $\psi \in \Psi_{\mathrm{AJ}}(G)$, and I be the subset of $\{1, \ldots, l\}$ consisting of i satisfying that in (5.5) the cuspidal representation π_i is self-dual and $\varepsilon(\pi_i) = -1$. Arthur's character $\varepsilon_\psi : \mathrm{C}_\psi \to \mu_2$ is defined by

$$\varepsilon_\psi(s) := \prod_{i \in I} \chi_i(s), \text{ for every } s \in \mathrm{C}_\psi.$$

The following result shows that it is sufficient to calculate the global epsilon factors $\varepsilon(\pi_i)$ for i in a subset of $\{1, \ldots, l\}$:

Proposition 5.6.4. *Let $\psi \in \Psi_{\mathrm{AJ}}(G)$. For any $s \in \mathrm{C}_\psi$, let I_s be the subset of $\{1,\dots,l\}$ consisting of i satisfying that in (5.5) the representation π_i is self-dual, d_i is even, and $\chi_i(s) = -1$. Then we have:*

$$\varepsilon_\psi(s) = \prod_{i \in I_s} \varepsilon(\pi_i).$$

Proof. When d_i is odd, the d_i-dimensional irreducible representation of $\mathrm{SL}_2(\mathbb{C})$ is orthogonal. Since the adjoint representation is an orthogonal representation, the self-dual representation π_i of $\mathcal{L}_{\mathbb{Z}}$ must be also orthogonal, which implies $\varepsilon(\pi_i) = 1$ by Theorem 5.4.8. Hence the subset I in Definition 5.6.3 is a subset of $\{i \,|\, d_i \text{ is even}\}$, and for any $s \in \mathrm{C}_\psi$ we have

$$\varepsilon_\psi(s) = \prod_{2|d_i,\, \pi_i = \pi_i^\vee,\, \varepsilon(\pi_i) = -1} \chi_i(s) = \prod_{2|d_i,\, \pi_i = \pi_i^\vee,\, \chi_i(s) = -1} \varepsilon(\pi_i) = \prod_{i \in I_s} \varepsilon(\pi_i).$$

□

5.6.3 The multiplicity formula

With two characters $\rho_\psi^\vee$ and ε_ψ in hand, we can state Arthur's following conjecture:

Conjecture 5.6.5. *(*Arthur's multiplicity formula*) Let G be a simply-connected anisotropic $\mathbb{Q}$-group with a reductive $\mathbb{Z}$-model, and π a level one adelic representation in $\Pi(G)$. We have the following formula for the multiplicity $\mathrm{m}(\pi)$ of π in the discrete spectrum $\mathcal{L}_{\mathrm{disc}}(G)$:*

$$\mathrm{m}(\pi) = \sum_{\psi \in \Psi_{\mathrm{disc}}(G),\, c(\psi) = c(\pi)} m_\psi, \quad \textit{where } m_\psi = \begin{cases} 1, & \textit{if } \rho_\psi^\vee = \varepsilon_\psi, \\ 0, & \textit{otherwise.} \end{cases} \tag{5.6}$$

6 Classification of global Arthur parameters for $\mathbf{F}_4$

In this section, we are going to apply Arthur's conjectures recalled in §5.5 and §5.6 to the simply-connected anisotropic $\mathbb{Q}$-group $\mathbf{F}_4$ defined in Definition 2.1.6. The dual group $\widehat{\mathbf{F}_4}$ is isomorphic to the extension $\mathbf{F}_{4,\mathbb{C}}$ of $\mathbf{F}_4$ to $\mathbb{C}$. In other words, the complex Lie group $\widehat{\mathbf{F}_4}(\mathbb{C})$ is isomorphic to the complexification $\mathrm{F}_{4,\mathbb{C}}$ of the real compact Lie group F_4.

6.1 Arthur parameters of $\mathbf{F}_4$

The real points $\mathrm{F}_4 = \mathbf{F}_4(\mathbb{R})$ is compact, so an adelic representation $\pi \in \Pi(\mathbf{F}_4)$ is determined uniquely by $\mathrm{c}(\pi)$. On the other hand, by Proposition 4.1.5 and Axiom 1, a discrete global Arthur parameter ψ of $\mathbf{F}_4$ is also determined uniquely by $\mathrm{c}(\psi) \in \mathcal{X}(\widehat{\mathbf{F}_4})$. Moreover, we have the following criterion, which is a direct corollary of Proposition 4.2.1:

Proposition 6.1.1. *Let ψ_1 and ψ_2 be two discrete global Arthur parameters of $\mathbf{F}_4$, and $\mathrm{r}_0 : \widehat{\mathbf{F}_4} \to \mathrm{SL}_{26,\mathbb{C}}$ the 26-dimensional irreducible representation of $\mathbf{F}_4(\mathbb{C})$. Then $\psi_1 = \psi_2$ if and only if $\mathrm{r}_0(\mathrm{c}(\psi_1)) = \mathrm{r}_0(\mathrm{c}(\psi_2))$.*

By this result, we will identify a discrete global Arthur parameter $\psi \in \Psi_{\mathrm{disc}}(\mathbf{F}_4)$ with the corresponding family of conjugacy classes $\mathrm{r}_0(\mathrm{c}(\psi)) \in \mathcal{X}(\mathrm{SL}_{26})$.

For a level one discrete automorphic representation $\pi \in \Pi_{\mathrm{disc}}(\mathbf{F}_4)$, the discrete global Arthur parameter $\psi \in \Psi_{\mathrm{AJ}}(\mathbf{F}_4)$ such that $\mathrm{c}(\psi) = \mathrm{c}(\pi)$ predicted by Axiom 1 is unique. We denote this parameter by ψ_π, which is identified with $\psi(\pi, \mathrm{r}_0) \in \mathcal{X}(\mathrm{SL}_{26})$. Conversely, for $\psi \in \Psi_{\mathrm{AJ}}(\mathbf{F}_4)$, we denote the unique representation $\pi \in \Pi(\pi)$ such that $\mathrm{c}(\pi) = \mathrm{c}(\psi)$ by π_ψ.

The following lemma gives us some constraint on the infinitesimal character $\mathrm{c}_\infty(\psi)$ of $\psi \in \Psi_{\mathrm{AJ}}(\mathbf{F}_4)$:

Lemma 6.1.2. *Let $\mathrm{c}_\infty \in (\mathfrak{f}_4)_{\mathrm{ss}}$ be the infinitesimal character of an irreducible representation of the compact group F_4, then there exists four non-negative integers a, b, c, d such that the eigenvalues (counted with multiplicity) of $\mathrm{r}_0(\mathrm{c}_\infty) \in (\mathfrak{sl}_{26})_{\mathrm{ss}}$ are:*

$$\begin{gathered}0, 0, \pm(a+1), \pm(b+1), \pm(a+b+2), \pm(b+c+2), \pm(a+b+c+3), \pm(b+c+d+3),\\ \pm(a+b+c+d+4), \pm(a+2b+c+4), \pm(a+2b+c+d+5), \pm(a+2b+2c+d+6),\\ \pm(a+3b+2c+d+7), \pm(2a+3b+2c+d+8).\end{gathered}$$

Proof. If we write the highest weight λ of this irreducible representation of F_4 as $a\varpi_1 + b\varpi_2 + c\varpi_3 + d\varpi_4$, then by Proposition 5.2.1 the infinitesimal character c_∞ is $\lambda + \rho = (a+1)\varpi_1 + (b+1)\varpi_2 + (c+1)\varpi_3 + (d+1)\varpi_4$. The eigenvalues of $\mathrm{r}_0(\mathrm{c}_\infty)$ are of the form $\langle \lambda + \rho, \alpha^\vee \rangle$, where $\alpha^\vee$ runs over the 26 weights of $\widehat{\mathbf{F}_4}(\mathbb{C})$ appearing in the representation r_0. By an easy calculation, we get the eigenvalues in the lemma. $\square$

As recalled in §5.3.1, we associate to $\psi \in \Psi_{\mathrm{AJ}}(\mathbf{F}_4)$ a morphism $\psi_{\mathrm{c}} : \mathcal{L}_{\mathbb{Z}} \times \mathrm{SU}(2) \to \mathrm{F}_4$ between compact Lie groups. This homomorphism inherits the following properties from ψ:

- the image $\mathrm{Im}(\psi_c)$ is connected due to Proposition 5.3.5,
- the centralizer of $\mathrm{Im}(\psi_{\mathrm{c}})$ in $\mathbf{F}_4$ coincides with the global component group C_ψ of ψ, which is an elementary finite abelian 2-group by [CL19, §8.4.14],
- and the zero weight appears exactly twice in the restriction of the 26-dimensional irreducible representation J_0 of F_4 along ψ_{c} by Lemma 6.1.2.

Hence $\mathrm{Im}(\psi_{\mathrm{c}})$ is a subgroup of F_4 satisfying the three conditions in the beginning of §4, thus the class $\mathrm{H}(\psi)$ defined in Definition 5.3.6 is the conjugacy class of one of the subgroups of F_4 listed in Theorem 4.6.7.

According to Conjecture 5.5.2, the discrete global Arthur parameter $\psi_\pi = \psi(\pi, \mathrm{r}_0)$ corresponding to a discrete automorphic representation $\pi \in \Pi_{\mathrm{disc}}(\mathbf{F}_4)$ should be of the form:

$$\pi_1[d_1] \oplus \cdots \oplus \pi_k[d_k],$$

where $\pi_i \in \Pi_{\mathrm{cusp}}(\mathrm{PGL}_{n_i})$ and $\sum_{i=1}^k n_i d_i = 26$. By Proposition 5.5.4, every π_i is algebraic, and it is also self-dual by the following lemma:

Lemma* 6.1.3. *Let $\pi \in \Pi_{\mathrm{disc}}(\mathbf{F}_4)$ and $\psi_\pi = \pi_1[d_1] \oplus \cdots \oplus \pi_k[d_k]$ be its corresponding discrete global Arthur parameter, then for each $i = 1, \ldots, k$, the representation $\pi_i \in \Pi_{\mathrm{cusp}}(\mathrm{PGL}_{n_i})$ is self-dual.*

Proof. By our classification result in §4.6, identifying $\pi_i \in \Pi_{\text{cusp}}(\mathrm{PGL}_{n_i})$ as an irreducible representation of $\mathcal{L}_{\mathbb{Z}}$, it must be of the form $\mathcal{L}_{\mathbb{Z}} \twoheadrightarrow H \xrightarrow{r} \mathrm{SL}_{n_i}(\mathbb{C})$, where H is a connected compact subgroup of F_4 and r is a self-dual irreducible representation of H, thus π_i itself is self-dual. $\square$

So a discrete global Arthur parameter $\psi \in \Psi_{\mathrm{AJ}}(\mathbf{F}_4)$ corresponding to some $\pi \in \Pi_{\text{disc}}(\mathbf{F}_4)$ must be of the form

$$\psi = \pi_1[d_1] \oplus \cdots \oplus \pi_k[d_k], \text{ where } \pi_i \in \Pi_{\text{alg}}^{\perp}(\mathrm{PGL}_{n_i}), \sum_{i=1}^{k} n_i d_i = 26. \tag{6.1}$$

The endoscopic types $(k, (n_i, d_i)_{1 \leq i \leq k})$ can be classified by our results in §4.6.

Example 6.1.4. If the class $\mathrm{H}(\psi)$ associated to $\psi \in \Psi_{\mathrm{AJ}}(\mathbf{F}_4)$ is the conjugacy class of

$$H = \left(\mathrm{A}_1^{[9,6^2,5]} \times \mathrm{A}_1^{[2^6,1^{14}]}\right)/\mu_2^{\Delta},$$

by §4.6.10 the restriction of the 26-dimensional irreducible representation (r_0, J_0) along ψ is isomorphic to

$$\mathrm{Sym}^5 \mathrm{St} \otimes \mathrm{St} + \mathrm{Sym}^8 \mathrm{St} \otimes \mathbf{1} + \mathrm{Sym}^4 \mathrm{St} \otimes \mathbf{1}.$$

Depending on how $\mathcal{L}_{\mathbb{Z}}$ and $\mathrm{SU}(2)$ are mapped to this subgroup $H \subset \mathrm{F}_4$, we have the following three possible endoscopic types for ψ:

- $(3, (2, 6), (1, 5), (1, 9))$, $\psi = \pi[6] \oplus [5] \oplus [9], \pi \in \Pi_{\text{alg}}^{\perp}(\mathrm{PGL}_2)$;
- $(3, (9, 1), (5, 1), (6, 2))$, $\psi = \mathrm{Sym}^8 \pi \oplus \mathrm{Sym}^4 \pi \oplus \mathrm{Sym}^5 \pi[2], \pi \in \Pi_{\text{alg}}^{\perp}(\mathrm{PGL}_2)$;
- $(3, (9, 1), (5, 1), (12, 1))$, $\psi = \mathrm{Sym}^8 \pi_1 \oplus \mathrm{Sym}^4 \pi_2 \oplus (\mathrm{Sym}^5 \pi_1 \otimes \pi_2), \pi_1, \pi_2 \in \Pi_{\text{alg}}^{\perp}(\mathrm{PGL}_2)$.

6.2 The multiplicity formula for $\mathbf{F}_4$

For a discrete global Arthur parameter $\psi \in \Psi_{\mathrm{AJ}}(\mathbf{F}_4)$, Arthur's multiplicity formula Conjecture 5.6.5 predicts that the multiplicity $\mathrm{m}(\pi_\psi)$ of π_ψ in $\mathcal{L}_{\text{disc}}(\mathbf{F}_4)$ equals to m_ψ, the formula for which is given in (5.6). To calculate m_ψ, it suffices to know two characters of C_ψ: Arthur's character ε_ψ, and $\rho_\psi^\vee$. We have given the formula of ε_ψ in Proposition 5.6.4, and in this subsection we will give a recipe for the character $\rho_\psi^\vee$ for our $\mathbb{Q}$-group $\mathbf{F}_4$.

We fix a maximal ideal $\widehat{T}$ of $\widehat{\mathbf{F}_4}$ and a Borel subgroup $\widehat{B} \supset \widehat{T}$ as in §5.6.1 such that the infinitesimal character $\mathrm{c}_\infty(\psi)$, as a cocharacter of $\widehat{T}$ is strictly dominant with respect to $(\widehat{\mathbf{F}_4}, \widehat{B}, \widehat{T})$. We denote the four simple roots of the root system with respect to $(\widehat{\mathbf{F}_4}, \widehat{B}, \widehat{T})$ by $\alpha_i^\vee, i = 1, 2, 3, 4$ [10].

By Lemma 6.1.2, we can order the eigenvalues (counted with multiplicity) of $\mathrm{c}_\infty(\psi)$ as $\mu_1 > \mu_2 > \mu_3 > \mu_4 > \mu_5 \geq \cdots > \mu_{26}$. The partial order relation of the positive weights of r_0 in Table 1 implies that

$$\mu_1 = \langle \mathrm{c}_\infty(\psi), 2\alpha_1^\vee + 3\alpha_2^\vee + 2\alpha_3^\vee + \alpha_4^\vee \rangle,\ \mu_4 = \langle \mathrm{c}_\infty(\psi), \alpha_1^\vee + 2\alpha_2^\vee + \alpha_3^\vee + \alpha_4^\vee \rangle.$$

[10] Here we still follow Bourbaki's notation, but since we are considering the root system of the dual group $\widehat{G}$, the simple root $\alpha_i^\vee, 1 \leq i \leq 4$ corresponds to α_{5-i} in Bourbaki.

Notice that

$$(2\alpha_1^\vee + 3\alpha_2^\vee + 2\alpha_3^\vee + \alpha_4^\vee) + (\alpha_1^\vee + 2\alpha_2^\vee + \alpha_3^\vee + \alpha_4^\vee) \equiv \alpha_1^\vee + \alpha_2^\vee + \alpha_3^\vee \equiv \rho_\psi^\vee \bmod 2X^*(\widehat{T}),$$

thus the character $\rho_\psi^\vee$ of $\mathrm{C}_\psi \subset \widehat{T}[2]$ is the product of $(2\alpha_1^\vee + 3\alpha_2^\vee + 2\alpha_3^\vee + \alpha_4^\vee)|_{\mathrm{C}_\psi}$ and $(\alpha_1^\vee + 2\alpha_2^\vee + \alpha_3^\vee + \alpha_4^\vee)|_{\mathrm{C}_\psi}$. Hence it suffices to determine these two characters.

If $\psi = \pi_1[d_1] \oplus \cdots \oplus \pi_k[d_k]$ as in (6.1), the eigenvalues of $\mathrm{r}_0(\mathrm{c}_\infty(\psi)) \in (\mathfrak{sl}_{26})_{\mathrm{ss}}$ are of the form $w + \frac{j}{2}$, where w is a weight of π_i and $j \in \{d_i - 1, d_i - 3, \ldots, -d_i + 3, -d_i + 1\}$. For each $i = 1, \ldots, k$, we define a multiset

$$\mathcal{W}_i := \left\{ w + \frac{j}{2} \,\middle|\, w \in \mathrm{Weights}(\pi_i) \text{ and } j = d_i - 1, d_i - 3, \ldots, -(d_i - 3), -(d_i - 1) \right\}.$$

Proposition 6.2.1. *There exists a unique index i (resp. j) in $\{1, \ldots, k\}$ such that $\mu_1 \in \mathcal{W}_i$ (resp. $\mu_4 \in \mathcal{W}_j$). If we denote respectively by ϵ_i and ϵ_j the characters of C_ψ induced by the C_ψ-actions on $\pi_i[d_i]$ and $\pi_j[d_j]$, then $\rho_\psi^\vee = \epsilon_i \cdot \epsilon_j$.*

Proof. The uniqueness of i and j follows from the fact that μ_1 and μ_4 are different from other eigenvalues of $\mathrm{r}_0(\mathrm{c}_\infty(\psi))$.

For any $s \in \mathrm{C}_\psi$, we have

$$\rho_\psi^\vee(s) = (2\alpha_1^\vee + 3\alpha_2^\vee + 2\alpha_3^\vee + \alpha_4^\vee)(s) \cdot (\alpha_1^\vee + 2\alpha_2^\vee + \alpha_3^\vee + \alpha_4^\vee)(s).$$

Since $\mu_1 \in \mathcal{W}_i$, the value $(2\alpha_1^\vee + 3\alpha_2^\vee + 2\alpha_3^\vee + \alpha_4^\vee)(s)$ is the scalar given by the action of s on the irreducible summand $\pi_i[d_i]$, which equals $\epsilon_i(s)$ by definition. Similarly, we have $(\alpha_1^\vee + 2\alpha_2^\vee + \alpha_3^\vee + \alpha_4^\vee)(s) = \epsilon_j(s)$ and the identity $\rho_\psi^\vee = \epsilon_i \cdot \epsilon_j$. ☐

6.3 Classification of Arthur parameters

Now we can do (*conjectural*) classification of global Arthur parameters for $\mathbf{F}_4$:

Theorem* 6.3.1. *Admitting the existence of the Langlands group $\mathcal{L}_\mathbb{Z}$ defined in §5.3 and Arthur's multiplicity formula Conjecture 5.6.5, a (level one) discrete global Arthur parameter $\psi \in \Psi_{\mathrm{AJ}}(\mathbf{F}_4)$ satisfies $\mathrm{m}(\pi_\psi) = 1$ if and only if it belongs to the parameters described in the following propositions (from Proposition 6.3.4 to Proposition 6.3.18).*

In this subsection, we will prove Theorem 6.3.1 case by case, depending on the conjugacy class $\mathrm{H}(\psi)$ associated to the discrete global Arthur parameter ψ. For each subgroup H of $\mathrm{F}_4 = \mathbf{F}_4(\mathbb{R})$ listed in §4.6, we classify all the endoscopic types of $\psi \in \Psi_{\mathrm{AJ}}(\mathbf{F}_4)$ such that $\mathrm{H}(\psi)$ is the conjugacy class of H like what we have done in Example 6.1.4, then apply Arthur's multiplicity formula Conjecture 5.6.5, Proposition 5.6.4 and Proposition 6.2.1 to ψ and get those with $\mathrm{m}(\pi_\psi) = 1$.

Notation 6.3.2. From now on, when $\mathrm{H}(\psi)$ is the F_4-conjugacy class of H, we say $\mathrm{H}(\psi) = H$ by an abuse of notation.

Remark 6.3.3. Since the proof of Theorem 6.3.1 is long, readers can read first the proof of Proposition 6.4.3 in §6.4 to see how Arthur's conjectures are used.

6.3.1 $H = \mathrm{A}_1^{[17,9]}$

The restriction of the 26-dimensional irreducible representation J_0 to H is isomorphic to

$$\mathrm{Sym}^{16}\,\mathrm{St} + \mathrm{Sym}^{8}\,\mathrm{St}.$$

For $\psi \in \Psi_{\mathrm{AJ}}(\mathbf{F}_4)$ satisfying $\mathrm{H}(\psi) = H$ and $\mathrm{m}(\pi_\psi) = 1$, there are two possible endoscopic types:

(i) $(2,(1,17),(1,9))$, which corresponds to the parameter $[17]\oplus[9]$ of the trivial representation of $\mathbf{F}_4(\mathbb{A})$.

(ii) $(2,(17,1),(9,1))$. The discrete global Arthur parameters ψ with this type are constructed as follows: for a representation $\pi \in \Pi_{\mathrm{alg}}^{\perp}(\mathrm{PGL}_2)$ and a positive integer k, we denote by $\mathrm{Sym}^k\,\pi$ the representation in $\Pi_{\mathrm{alg,reg}}^{\perp}(\mathrm{PGL}_{k+1})$ corresponding to the irreducible representation given by

$$\mathcal{L}_{\mathbb{Z}} \xrightarrow{\psi_\pi} \mathrm{SL}_2(\mathbb{C}) \to \mathrm{SL}(\mathrm{Sym}^k\,\mathrm{St}) \simeq \mathrm{SL}_{k+1}(\mathbb{C}).$$

A global Arthur parameter of this type is of the form:

$$\mathrm{Sym}^{16}\,\pi \oplus \mathrm{Sym}^{8}\,\pi,\ \pi \in \Pi_{\mathrm{alg}}^{\perp}(\mathrm{PGL}_2).$$

Proposition* 6.3.4. *For a discrete global Arthur parameter $\psi \in \Psi_{\mathrm{AJ}}(\mathbf{F}_4)$ satisfying $\mathrm{H}(\psi) = H$, the multiplicity $\mathrm{m}(\pi_\psi) = 1$ if and only if ψ is one of the following parameters:*

- $[17]\oplus[9]$*, which corresponds to the trivial representation of* $\mathbf{F}_4(\mathbb{A})$.
- $\mathrm{Sym}^{16}\,\pi \oplus \mathrm{Sym}^{8}\,\pi,\ \pi \in \Pi_{\mathrm{alg}}^{\perp}(\mathrm{PGL}_2)$.

Proof. This is because C_ψ is trivial. □

6.3.2 $H = \left(\mathrm{A}_1^{[9,6^2,5]} \times \mathrm{A}_1^{[2^6,1^{14}]}\right)/\mu_2^{\Delta}$

By §4.6.10 the restriction of the 26-dimensional irreducible representation J_0 of F_4 to H is isomorphic to

$$\mathrm{Sym}^5\,\mathrm{St}\otimes\mathrm{St} + (\mathrm{Sym}^8\,\mathrm{St} + \mathrm{Sym}^4\,\mathrm{St})\otimes\mathbf{1},$$

and the centralizer of H in F_4 is $\mathrm{Z}(H) \simeq \mathbb{Z}/2\mathbb{Z}$.

For $\psi \in \Psi_{\mathrm{AJ}}(\mathbf{F}_4)$ satisfying $\mathrm{H}(\psi) = H$ and $\mathrm{m}(\pi_\psi) = 1$, there are three possible endoscopic types:

(i) $(3,(2,6),(1,5),(1,9))$. A global Arthur parameter of this type is of the form:

$$\pi[6]\oplus[5]\oplus[9],\ \pi \in \Pi_{\mathrm{alg}}^{\perp}(\mathrm{PGL}_2).$$

(ii) $(3,(9,1),(5,1),(6,2))$. A global Arthur parameter of this type is of the form:

$$\mathrm{Sym}^8\,\pi \oplus \mathrm{Sym}^4\,\pi \oplus \mathrm{Sym}^5\,\pi[2],\ \pi \in \Pi_{\mathrm{alg}}^{\perp}(\mathrm{PGL}_2).$$

(iii) $(3,(12,1),(9,1),(5,1))$. For two representations $\pi_1,\pi_2\in\Pi_{\mathrm{alg}}^{\perp}(\mathrm{PGL}_2)$, we can construct the following 12-dimensional irreducible representation of $\mathcal{L}_{\mathbb{Z}}$:

$$\mathcal{L}_{\mathbb{Z}}\xrightarrow{(\psi_{\pi_1},\psi_{\pi_2})}\mathrm{SL}_2(\mathbb{C})\times\mathrm{SL}_2(\mathbb{C})\xrightarrow{\mathrm{Sym}^5\otimes\mathrm{id}}\mathrm{SL}_{12}(\mathbb{C}),$$

which induces a cuspidal representation of PGL_{12}, denoted by $\mathrm{Sym}^5\pi_1\otimes\pi_2$. A global Arthur parameter of this type is of the form:

$$\mathrm{Sym}^8\pi_1\oplus\mathrm{Sym}^4\pi_1\oplus\left(\mathrm{Sym}^5\pi_1\otimes\pi_2\right),\ \pi_1,\pi_2\in\Pi_{\mathrm{alg}}^{\perp}(\mathrm{PGL}_2).$$

Remark 6.3.5. In fact, for a $(3,(12,1),(9,1),(5,1))$-type parameter

$$\psi=\mathrm{Sym}^8\pi_1\oplus\mathrm{Sym}^4\pi_1\oplus\left(\mathrm{Sym}^5\pi_1\otimes\pi_2\right),\ \pi_1,\pi_2\in\Pi_{\mathrm{alg}}^{\perp}(\mathrm{PGL}_2),$$

there are some conditions on the motivic weights $\mathrm{w}(\pi_1),\mathrm{w}(\pi_2)$ to make ψ a parameter in $\Psi_{\mathrm{AJ}}(\mathbf{F}_4)$. We will add these conditions for global Arthur parameters ψ with $m_\psi=1$ when necessary. For example, when $\mathrm{w}(\pi_2)>9\mathrm{w}(\pi_1)$ the condition for $\psi\in\Psi(\mathbf{F}_4)$ is that $\mathrm{w}(\pi_2)\geq 9\mathrm{w}(\pi_1)+2$, which is satisfied automatically since $\mathrm{w}(\pi_2)$ and $9\mathrm{w}(\pi_1)$ are two distinct odd numbers.

For this subgroup H of F_4, the restriction of the adjoint representation $\mathfrak{f}_4$ of F_4 to H is isomorphic to

$$\mathbf{1}\otimes\mathrm{Sym}^2\mathrm{St}+\left(\mathrm{Sym}^9\mathrm{St}+\mathrm{Sym}^3\mathrm{St}\right)\otimes\mathrm{St}+\left(\mathrm{Sym}^{10}\mathrm{St}+\mathrm{Sym}^6\mathrm{St}+\mathrm{Sym}^2\mathrm{St}\right)\otimes\mathbf{1}.$$

Proposition* 6.3.6. *For a discrete global Arthur parameter $\psi\in\Psi_{\mathrm{AJ}}(\mathbf{F}_4)$ satisfying $\mathrm{H}(\psi)=H$, the multiplicity $\mathrm{m}(\pi_\psi)=1$ if and only if ψ is one of the following parameters:*

- $\pi[6]\oplus[5]\oplus[9]$, *where* $\pi\in\Pi_{\mathrm{alg}}^{\perp}(\mathrm{PGL}_2)$.
- $\mathrm{Sym}^8\pi\oplus\mathrm{Sym}^4\pi\oplus\mathrm{Sym}^5\pi[2]$, *where* $\pi\in\Pi_{\mathrm{alg}}^{\perp}(\mathrm{PGL}_2)$ *satisfies* $\mathrm{w}(\pi)\equiv 3\bmod 4$.
- $\mathrm{Sym}^8\pi_1\oplus\mathrm{Sym}^4\pi_1\oplus\left(\mathrm{Sym}^5\pi_1\otimes\pi_2\right)$, *where* $\pi_1,\pi_2\in\Pi_{\mathrm{alg}}^{\perp}(\mathrm{PGL}_2)$ *have motivic weights* w_1,w_2 *respectively such that* $w_2>9w_1$ *or* $5w_1<w_2<7w_1$.

Proof. We denote the generator of $\mathrm{C}_\psi=\mathrm{Z}(H)$ by γ.

Case (i): $\psi=\pi[6]\oplus[5]\oplus[9]$, where $\pi\in\Pi_{\mathrm{alg}}^{\perp}(\mathrm{PGL}_2)$ has motivic weight w. In this case the restriction of $\mathfrak{f}_4$ along ψ is isomorphic to

$$\mathrm{Sym}^2\pi\oplus\pi[10]\oplus\pi[4]\oplus[11]\oplus[7]\oplus[3].$$

By Proposition 5.6.4, we have:

$$\varepsilon_\psi(\gamma)=\varepsilon(\pi)\cdot\varepsilon(\pi)=\varepsilon(\mathbf{I}_w)^2=1.$$

On the other side, since $w\geq 11$ we have $\mu_1=\frac{w+5}{2}$ and $\mu_4=\frac{w-1}{2}$. Both of them come from the irreducible summand $\pi[6]$ in ψ, so $\rho_\psi^\vee$ must be the trivial character by Proposition 6.2.1. By Arthur's multiplicity formula, $\mathrm{m}(\pi_\psi)=1$ for any $\pi\in\Pi_{\mathrm{alg}}^{\perp}(\mathrm{PGL}_2)$.

Case (ii): $\psi=\mathrm{Sym}^8\pi\oplus\mathrm{Sym}^4\pi\oplus\mathrm{Sym}^5\pi[2]$, where $\pi\in\Pi_{\mathrm{alg}}^{\perp}(\mathrm{PGL}_2)$ has motivic weight w. In this case the restriction of $\mathfrak{f}_4$ along ψ is isomorphic to

$$\mathrm{Sym}^{10}\pi\oplus\mathrm{Sym}^9\pi[2]\oplus\mathrm{Sym}^6\pi\oplus\mathrm{Sym}^3\pi[2]\oplus\mathrm{Sym}^2\pi\oplus[3].$$

By Proposition 5.6.4, we have:

$$\begin{aligned}
\varepsilon_\psi(\gamma) &= \varepsilon(\mathrm{Sym}^3\pi)\cdot\varepsilon(\mathrm{Sym}^9\pi)\\
&= \varepsilon(\mathbf{I}_{3w}+\mathbf{I}_w)\cdot\varepsilon(\mathbf{I}_{9w}+\mathbf{I}_{7w}+\mathbf{I}_{5w}+\mathbf{I}_{3w}+\mathbf{I}_w)\\
&= (-1)^{(w+1)/2+(3w+1)/2}\cdot(-1)^{(w+1)/2+(3w+1)/2+(5w+1)/2+(7w+1)/2+(9w+1)/2}\\
&= (-1)^{(w+3)/2}.
\end{aligned}$$

On the other side, $\mu_1 = 4w$ comes from $\mathrm{Sym}^8\pi$ and $\mu_4 = \frac{5w-1}{2}$ comes from $\mathrm{Sym}^5\pi[2]$. So $\rho_\psi^\vee(\gamma) = -1$ by Proposition 6.2.1. By Arthur's multiplicity formula, $\mathrm{m}(\pi_\psi) = 1$ if and only if $w \equiv 3 \bmod 4$.

Case (iii): $\psi = \mathrm{Sym}^8\pi_1 \oplus \mathrm{Sym}^4\pi_1 \oplus \left(\mathrm{Sym}^5\pi_1\otimes\pi_2\right)$, where $\pi_1,\pi_2 \in \Pi_{\mathrm{alg}}^{\perp}(\mathrm{PGL}_2)$ have motivic weight w_1, w_2 respectively. Since this parameter is tempered, the character ε_ψ is always trivial. We only need to find what condition w_1, w_2 should satisfy to make $\rho_\psi^\vee(\gamma) = 1$. In this case, γ acts on $\mathrm{Sym}^8\pi_1$ and $\mathrm{Sym}^4\pi_1$ by 1 and on $\mathrm{Sym}^5\pi_1\otimes\pi_2$ by -1. We can see that $\mu_1 = 4w_1$ or $\frac{5w_1+w_2}{2}$, depending on the values of w_1, w_2.

(1) If $\mu_1 = 4w_1$, which is equivalent to $w_2 < 3w_1$. Now $\rho_\psi^\vee(\gamma) = 1$ if and only if $\mu_4 = 3w_1$ since the other positive weights $w_1, 2w_1$ in $\mathrm{Sym}^4\pi_1 \oplus \mathrm{Sym}^8\pi_1$ both have multiplicity 2. However, $3w_1$ is larger than all the Hodge weights of ψ except $4w_1$ and $\frac{5w_1+w_2}{2}$, which shows that it can only be μ_2 or μ_3. So in this case $\rho_\psi^\vee(\gamma) = -1$.

(2) If $\mu_1 = \frac{5w_1+w_2}{2}$, which is equivalent to $w_2 > 3w_1$. Now $\rho_\psi^\vee(\gamma) = 1$ if and only if $\mu_4 = \frac{w_1+w_2}{2}$ or $\frac{-w_1+w_2}{2}$.
   (a) $\mu_4 = \frac{w_1+w_2}{2}$ is equivalent to $4w_1 > \frac{w_1+w_2}{2} > 3w_1$, thus $5w_1 < w_2 < 7w_1$.
   (b) $\mu_4 = \frac{-w_1+w_2}{2}$ is equivalent to $\frac{-w_1+w_2}{2} > 4w_1$, thus $w_2 > 9w_1$.

By Arthur's multiplicity formula $\mathrm{m}(\pi_\psi) = 1$ if and only if $w_2 > 9w_1$ or $5w_1 < w_2 < 7w_1$. $\square$

6.3.3 $H = \left(\mathrm{A}_1^{[5,4^2,3^3,2^2]}\times\mathrm{A}_1^{[3^3,2^6,1^5]}\right)/\mu_2^{\Delta}$

By §4.6.11 the restriction of the 26-dimensional irreducible representation J_0 of F_4 to H is isomorphic to

$$\mathrm{Sym}^4\mathrm{St}\otimes\mathbf{1} + \left(\mathrm{Sym}^3\mathrm{St}+\mathrm{St}\right)\otimes\mathrm{St} + \mathrm{Sym}^2\mathrm{St}\otimes\mathrm{Sym}^2\mathrm{St},$$

and the centralizer of H in F_4 is $\mathrm{Z}(H) \simeq \mathbb{Z}/2\mathbb{Z}$.

For $\psi \in \Psi_{\mathrm{AJ}}(\mathbf{F}_4)$ satisfying $\mathrm{H}(\psi) = H$ and $\mathrm{m}(\pi_\psi) = 1$, there are three possible endoscopic types:

(i) $(4,(3,3),(2,4),(2,2),(1,5))$. A global Arthur parameter of this type is of the form:

$$\mathrm{Sym}^2\pi[3]\oplus\pi[4]\oplus\pi[2]\oplus[5],\ \pi\in\Pi_{\mathrm{alg}}^{\perp}(\mathrm{PGL}_2).$$

(ii) $(4,(5,1),(4,2),(3,3),(2,2))$. A global Arthur parameter of this type is of the form:

$$\mathrm{Sym}^4\pi\oplus\mathrm{Sym}^3\pi[2]\oplus\mathrm{Sym}^2\pi[3]\oplus\pi[2],\ \pi\in\Pi_{\mathrm{alg}}^{\perp}(\mathrm{PGL}_2).$$

(iii) $(4,(9,1),(8,1),(5,1),(4,1))$. A global Arthur parameter of this type is of the form:

$$\mathrm{Sym}^4\pi_1 \oplus (\mathrm{Sym}^3\pi_1\otimes\pi_2)\oplus(\mathrm{Sym}^2\pi_1\otimes\mathrm{Sym}^2\pi_2)\oplus(\pi_1\otimes\pi_2),\ \pi_1,\pi_2\in\Pi_{\mathrm{alg}}^{\perp}(\mathrm{PGL}_2),$$

where the representations $\mathrm{Sym}^k\pi_1\otimes\mathrm{Sym}^l\pi_2$ are defined similarly as the representation $\mathrm{Sym}^5\pi_1\otimes\pi_2$ appearing in $[(12,1),(9,1),(5,1)]$-type parameters introduced in §6.3.2.

For this subgroup H of F_4, the restriction of the adjoint representation $\mathfrak{f}_4$ of F_4 to H is isomorphic to

$$\mathrm{St}\otimes\mathrm{Sym}^3\mathrm{St}+\left(\mathrm{Sym}^4\mathrm{St}+\mathbf{1}\right)\otimes\mathrm{Sym}^2\mathrm{St}+\left(\mathrm{Sym}^5\mathrm{St}+\mathrm{Sym}^3\mathrm{St}\right)\otimes\mathrm{St}+\left(\mathrm{Sym}^2\mathrm{St}\right)^{\oplus 2}\otimes\mathbf{1}.$$

Proposition* 6.3.7. *For a discrete global Arthur parameter $\psi\in\Psi_{\mathrm{AJ}}(\mathbf{F}_4)$ satisfying $\mathrm{H}(\psi)=H$, the multiplicity $\mathrm{m}(\pi_\psi)=1$ if and only if ψ is one of the following parameters:*

- $\mathrm{Sym}^2\pi[3]\oplus\pi[4]\oplus\pi[2]\oplus[5]$, *where* $\pi\in\Pi_{\mathrm{alg}}^{\perp}(\mathrm{PGL}_2)$.
- $\mathrm{Sym}^4\pi\oplus\mathrm{Sym}^3\pi[2]\oplus\mathrm{Sym}^2\pi[3]\oplus\pi[2]$, *where* $\pi\in\Pi_{\mathrm{alg}}^{\perp}(\mathrm{PGL}_2)$.
- $\mathrm{Sym}^4\pi_1\oplus(\mathrm{Sym}^3\pi_1\otimes\pi_2)\oplus(\mathrm{Sym}^2\pi_1\otimes\mathrm{Sym}^2\pi_2)\oplus(\pi_1\otimes\pi_2)$, *where* $\pi_1,\pi_2\in\Pi_{\mathrm{alg}}^{\perp}(\mathrm{PGL}_2)$ *have motivic weights w_1,w_2 respectively such that*

$$w_1>3w_2 \text{ or } w_1<w_2<3w_1 \text{ or } 3w_1<w_2<5w_1.$$

Proof. We denote the generator of $\mathrm{C}_\psi=\mathrm{Z}(H)$ by γ.

Case (i): $\psi=\mathrm{Sym}^2\pi[3]\oplus\pi[4]\oplus\pi[2]\oplus[5]$, where $\pi\in\Pi_{\mathrm{alg}}^{\perp}(\mathrm{PGL}_2)$ has motivic weight w. In this case the restriction of $\mathfrak{f}_4$ along ψ is isomorphic to

$$\mathrm{Sym}^3\pi[2]\oplus\mathrm{Sym}^2\pi[5]\oplus\mathrm{Sym}^2\pi\oplus\pi[6]\oplus\pi[4]\oplus[3]\oplus[3].$$

By Proposition 5.6.4, we have:

$$\varepsilon_\psi(\gamma)=\varepsilon(\mathrm{Sym}^3\pi)\cdot\varepsilon(\pi)\cdot\varepsilon(\pi)=\varepsilon(\mathbf{I}_{3w}+\mathbf{I}_w)\cdot\varepsilon(\mathbf{I}_w)^2=(-1)^{2w+1}=-1.$$

On the other side, $\mu_1=w+1$ comes from $\mathrm{Sym}^2\pi[3]$ and $\mu_4=\frac{w+3}{2}$ comes from $\pi[4]$. Since γ acts on $\mathrm{Sym}^2\pi[3]$ by 1 and on $\pi[4]$ by -1, we have $\rho_\psi^\vee(\gamma)=1$ by Proposition 6.2.1. By Arthur's multiplicity formula, $\mathrm{m}(\pi_\psi)=1$ for any $\pi\in\Pi_{\mathrm{alg}}^{\perp}(\mathrm{PGL}_2)$.

Case (ii): $\psi=\mathrm{Sym}^4\pi\oplus\mathrm{Sym}^3\pi[2]\oplus\mathrm{Sym}^2\pi[3]\oplus\pi[2]$, where $\pi\in\Pi_{\mathrm{alg}}^{\perp}(\mathrm{PGL}_2)$ has motivic weight w. In this case the restriction of $\mathfrak{f}_4$ along ψ is isomorphic to

$$\mathrm{Sym}^5\pi[2]\oplus\mathrm{Sym}^4\pi[3]\oplus\mathrm{Sym}^3\pi[2]\oplus(\mathrm{Sym}^2\pi)^{\oplus 2}\oplus\pi[4]\oplus[3].$$

By Proposition 5.6.4, we have:

$$\varepsilon_\psi(\gamma)=\varepsilon(\pi)\cdot\varepsilon(\mathrm{Sym}^3\pi)\cdot\varepsilon(\mathrm{Sym}^5\pi)=\varepsilon(\mathbf{I}_w)\varepsilon(\mathbf{I}_{3w}+\mathbf{I}_w)\varepsilon(\mathbf{I}_{5w}+\mathbf{I}_{3w}+\mathbf{I}_w)=(-1)^{3w+1}=1.$$

On the other side, $\mu_1=2w$ comes from $\mathrm{Sym}^4\pi$ and $\mu_4=w+1$ comes from $\mathrm{Sym}^2\pi[3]$. Since γ acts on $\mathrm{Sym}^4\pi$ and $\mathrm{Sym}^2\pi[3]$ both by 1, we have $\rho_\psi^\vee(\gamma)=1$ by Proposition 6.2.1. Arthur's multiplicity formula shows that $\mathrm{m}(\pi_\psi)=1$ for any $\pi\in\Pi_{\mathrm{alg}}^{\perp}(\mathrm{PGL}_2)$.

Case (iii): $\psi=\mathrm{Sym}^4\pi_1\oplus(\mathrm{Sym}^3\pi_1\otimes\pi_2)\oplus(\mathrm{Sym}^2\pi_1\otimes\mathrm{Sym}^2\pi_2)\oplus(\pi_1\otimes\pi_2)$, where $\pi_1,\pi_2\in\Pi_{\mathrm{alg}}^{\perp}(\mathrm{PGL}_2)$ have motivic weights w_1,w_2 respectively. The motivic weights satisfy $w_2\neq w_1, w_2\neq 3w_1$, otherwise the zero weight appears more than twice and ψ fails to be in $\Psi_{\mathrm{AJ}}(\mathbf{F}_4)$. In this case ε_ψ is trivial. The element γ acts on $\mathrm{Sym}^4\pi_1$ and $\mathrm{Sym}^2\pi_1\otimes\mathrm{Sym}^2\pi_2$ by 1, and on $\mathrm{Sym}^3\pi_1\otimes\pi_2, \pi_1\otimes\pi_2$ by -1. The largest weight μ_1 is $2w_1$ or w_1+w_2.

(1) If $w_1 > w_2$, then $\mu_1 = 2w_1$. Now μ_4 equals to $\frac{3w_1-w_2}{2}$ or $w_1 + w_2$. The character $\rho_\psi^\vee$ is trivial if and only if $\mu_4 = w_1 + w_2$, which is equivalent to $w_1 > 3w_2$.

(2) If $w_1 < w_2$, then $\mu_1 = w_1 + w_2$.

(a) If $w_2 > 3w_1$, then

$$w_1 + w_2 > w_2 > \max(-w_1 + w_2, \frac{3w_1 + w_2}{2}) > \min(-w_1 + w_2, \frac{3w_1 + w_2}{2})$$

and they are larger than other weights, thus $\mu_4 = -w_1 + w_2$ or $\frac{3w_1+w_2}{2}$. So $\rho_\psi^\vee(\gamma) = 1$ if and only if $\mu_4 = -w_1 + w_2$, thus if and only if $\frac{3w_1+w_2}{2} > w_2 - w_1$, which is equivalent to that $3w_1 < w_2 < 5w_1$.

(b) If $w_2 < 3w_1$, then

$$w_1 + w_2 > \frac{3w_1 + w_2}{2} > \max(2w_1, w_2) > \min(2w_1, w_2)$$

and they are larger than other weights. So we always have $\rho_\psi^\vee(\gamma) = 1$.

By Arthur's multiplicity formula, $\mathrm{m}(\pi_\psi) = 1$ if and only if $w_1 > 3w_2$ or $w_1 < w_2 < 5w_1$ and $w_2 \neq 3w_1$. □

6.3.4 $H = \left(\mathrm{A}_1^{[4^2,3^3,2^4,1]} \times \mathrm{A}_1^{[4^2,3^3,2^4,1]}\right)/\mu_2^\Delta$

By §4.6.12, the restriction of the 26-dimensional irreducible representation J_0 of F_4 to H is isomorphic to

$$\mathbf{1} + \mathrm{Sym}^3 \mathrm{St} \otimes \mathrm{St} + \mathrm{Sym}^2 \mathrm{St} \otimes \mathrm{Sym}^2 \mathrm{St} + \mathrm{St} \otimes \mathrm{Sym}^3 \mathrm{St},$$

and the centralizer of H in F_4 is $\mathrm{Z}(H) \simeq \mathbb{Z}/2\mathbb{Z}$.

For $\psi \in \Psi_{\mathrm{AJ}}(\mathbf{F}_4)$ satisfying $\mathrm{H}(\psi) = H$ and $\mathrm{m}(\pi_\psi) = 1$, there are two possible endoscopic types:

(i) $(4, (4, 2), (3, 3), (2, 4), (1, 1))$. A global Arthur parameter of this type is of the form:

$$\mathrm{Sym}^3 \pi[2] \oplus \mathrm{Sym}^2 \pi[3] \oplus \pi[4] \oplus [1],\ \pi \in \Pi_{\mathrm{alg}}^\perp(\mathrm{PGL}_2).$$

(ii) $(4, (9, 1), (8, 1), (8, 1), (1, 1))$. A global Arthur parameter of this type is of the form:

$$(\mathrm{Sym}^3 \pi_1 \otimes \pi_2) \oplus (\mathrm{Sym}^2 \pi_1 \otimes \mathrm{Sym}^2 \pi_2) \oplus (\pi_1 \otimes \mathrm{Sym}^3 \pi_2) \oplus [1],\ \pi_1, \pi_2 \in \Pi_{\mathrm{alg}}^\perp(\mathrm{PGL}_2).$$

For this subgroup H of F_4, the restriction of the adjoint representation $\mathfrak{f}_4$ of F_4 to H is isomorphic to

$$\left(\mathrm{Sym}^4 \mathrm{St} + \mathbf{1}\right) \otimes \mathrm{Sym}^2 \mathrm{St} + \mathrm{Sym}^2 \mathrm{St} \otimes \left(\mathrm{Sym}^4 \mathrm{St} + \mathbf{1}\right) + \mathrm{Sym}^3 \mathrm{St} \otimes \mathrm{St} + \mathrm{St} \otimes \mathrm{Sym}^3 \mathrm{St}.$$

Proposition* 6.3.8. *A discrete global Arthur parameter $\psi \in \Psi_{\mathrm{AJ}}(\mathbf{F}_4)$ satisfying $\mathrm{H}(\psi) = H$ and $\mathrm{m}(\pi_\psi) = 1$ must be of one of the following parameters:*

- $\mathrm{Sym}^3 \pi[2] \oplus \mathrm{Sym}^2 \pi[3] \oplus \pi[4] \oplus [1]$, *where $\pi \in \Pi_{\mathrm{alg}}^\perp(\mathrm{PGL}_2)$ satisfies $\mathrm{w}(\pi) \equiv 3 \bmod 4$.*

- $(\mathrm{Sym}^3\pi_1\otimes\pi_2)\oplus(\mathrm{Sym}^2\pi_1\otimes\mathrm{Sym}^2\pi_2)\oplus(\pi_1\otimes\mathrm{Sym}^3\pi_2)\oplus[1]$, *where* π_1,π_2 *have motivic weights* w_1,w_2 *respectively such that* $w_2<w_1<3w_2$.

Proof. We denote the generator of $\mathrm{C}_\psi=\mathrm{Z}(H)$ by σ.

Case (i): $\psi=\mathrm{Sym}^3\pi[2]\oplus\mathrm{Sym}^2\pi[3]\oplus\pi[4]\oplus[1]$, where $\pi\in\Pi_{\mathrm{alg}}^{\perp}(\mathrm{PGL}_2)$ has motivic weight w. In this case the restriction of $\mathfrak{f}_4$ along ψ is isomorphic to

$$\mathrm{Sym}^4\pi[3]\oplus\mathrm{Sym}^3\pi[2]\oplus\mathrm{Sym}^2\pi[5]\oplus\mathrm{Sym}^2\pi\oplus\pi[4]\oplus[3].$$

By Proposition 5.6.4, we have:

$$\varepsilon_\psi(\sigma)=\varepsilon(\mathrm{Sym}^3\pi)\cdot\varepsilon(\pi)=\varepsilon(\mathbf{I}_{3w}+\mathbf{I}_w)\cdot\varepsilon(\mathbf{I}_w)=(-1)^{(3w+1)/2}.$$

On the other side, $\mu_1=\frac{3w+1}{2}$ comes from $\mathrm{Sym}^3\pi[2]$ and $\mu_4=w$ comes from $\mathrm{Sym}^2\pi[3]$. Since σ acts on $\mathrm{Sym}^3\pi[2]$ by -1 and on $\mathrm{Sym}^2\pi[3]$ by 1, we have $\rho_\psi^\vee(\sigma)=-1$ by Proposition 6.2.1. By Arthur's multiplicity formula, $\mathrm{m}(\pi_\psi)=1$ if and only if $w\equiv 3 \bmod 4$.

Case (ii): $\psi=(\mathrm{Sym}^3\pi_1\otimes\pi_2)\oplus(\mathrm{Sym}^2\pi_1\otimes\mathrm{Sym}^2\pi_2)\oplus(\pi_1\otimes\mathrm{Sym}^3\pi_2)\oplus[1]$, where $\pi_1,\pi_2\in\Pi_{\mathrm{alg}}^{\perp}(\mathrm{PGL}_2)$ have motivic weights $w_1>w_2$ respectively. In this case, ε_ψ is trivial. On the other side, $\mu_1=\frac{3w_1+w_2}{2}$ and $\mu_4=w_1$ or $\frac{w_1+3w_2}{2}$ or $\frac{3w_1-w_2}{2}$. By Proposition 6.2.1, $\rho_\psi^\vee$ is trivial if and only if $\mu_4=\frac{w_1+3w_2}{2}$ or $\frac{3w_1-w_2}{2}$.

(1) $\mu_4=\frac{w_1+3w_2}{2}$ if and only if $\frac{3w_1-w_2}{2}>\frac{w_1+3w_2}{2}>w_1$, which is equivalent to $2w_2<w_1<3w_2$.

(2) $\mu_4=\frac{3w_1-w_2}{2}$ if and only if $\frac{w_1+3w_2}{2}>\frac{3w_1-w_2}{2}$, which is equivalent to $w_1<2w_2$.

By Arthur's multiplicity formula, $\mathrm{m}(\pi_\psi)=1$ if and only if $w_2<w_1<3w_2$ and $w_1\neq 2w_2$. Notice that $w_1\neq 2w_2$ holds automatically since w_1 is odd. ☐

6.3.5 $H=\mathrm{A}_1^{[7^3,1^5]}\times\mathrm{A}_1^{[5,3^7]}$

By §4.6.7, the restriction of the 26-dimensional irreducible representation J_0 of F_4 to H is isomorphic to

$$\mathrm{Sym}^6\mathrm{St}\otimes\mathrm{Sym}^2\mathrm{St}+\mathbf{1}\otimes\mathrm{Sym}^4\mathrm{St},$$

and the centralizer of H in F_4 is trivial.

For $\psi\in\Psi_{\mathrm{AJ}}(\mathbf{F}_4)$ satisfying $\mathrm{H}(\psi)=H$ and $\mathrm{m}(\pi_\psi)=1$, there are three possible endoscopic types:

(i) $(2,(7,3),(1,5))$. A global Arthur parameter of this type is of the form:

$$\mathrm{Sym}^6\pi[3]\oplus[5],\ \pi\in\Pi_{\mathrm{alg}}^{\perp}(\mathrm{PGL}_2).$$

(ii) $(2,(5,1),(3,7))$. A global Arthur parameter of this type is of the form:

$$\mathrm{Sym}^4\pi\oplus\mathrm{Sym}^2\pi[7],\ \pi\in\Pi_{\mathrm{alg}}^{\perp}(\mathrm{PGL}_2).$$

(iii) $(2,(21,1),(5,1))$. A global Arthur parameter of this type is of the form:

$$\big(\mathrm{Sym}^6\pi_1\otimes\mathrm{Sym}^2\pi_2\big)\oplus\mathrm{Sym}^4\pi_2,\ \pi_1,\pi_2\in\Pi_{\mathrm{alg}}^{\perp}(\mathrm{PGL}_2).$$

Proposition* 6.3.9. *A discrete global Arthur parameter $\psi \in \Psi_{\mathrm{AJ}}(\mathbf{F}_4)$ satisfying $\mathrm{H}(\psi) = H$ and $\mathrm{m}(\pi_\psi) = 1$ must be of one of the following parameters:*

- $\mathrm{Sym}^6 \pi[3] \oplus [5]$, *where* $\pi \in \Pi^{\perp}_{\mathrm{alg}}(\mathrm{PGL}_2)$.
- $\mathrm{Sym}^4 \pi \oplus \mathrm{Sym}^2 \pi[7]$, *where* $\pi \in \Pi^{\perp}_{\mathrm{alg}}(\mathrm{PGL}_2)$.
- $\left(\mathrm{Sym}^6 \pi_1 \otimes \mathrm{Sym}^2 \pi_2\right) \oplus \mathrm{Sym}^4 \pi_2$, *where* $\pi_1, \pi_2 \in \Pi^{\perp}_{\mathrm{alg}}(\mathrm{PGL}_2)$ *have motivic weights* w_1, w_2 *respectively such that* $w_2 \neq w_1$ *and* $w_2 \neq 3w_1$.

Proof. This follows from the fact that C_ψ is trivial. The conditions $w_2 \neq w_1$ and $w_2 \neq 3w_1$ in the third case are equivalent to that $\psi = \left(\mathrm{Sym}^6 \pi_1 \otimes \mathrm{Sym}^2 \pi_2\right) \oplus \mathrm{Sym}^4 \pi_2 \in \Psi_{\mathrm{AJ}}(\mathbf{F}_4)$. ☐

6.3.6 $H = \mathrm{A}_1^{[5,3^7]} \times \left(\mathrm{A}_1^{[3^3,2^6,1^5]} \times \mathrm{A}_1^{[2^6,1^{14}]}\right)/\mu_2^{\Delta}$

By §4.6.8, the restriction of the 26-dimensional irreducible representation J_0 of F_4 to H is isomorphic to

$$\mathrm{Sym}^4 \mathrm{St} \otimes \mathbf{1} \otimes \mathbf{1} + \mathrm{Sym}^2 \mathrm{St} \otimes \left(\mathrm{St} \otimes \mathrm{St} + \mathrm{Sym}^2 \mathrm{St} \otimes \mathbf{1}\right),$$

and the centralizer of H in F_4 is $\mathrm{Z}(H) \simeq \mathbb{Z}/2\mathbb{Z}$.

For $\psi \in \Psi_{\mathrm{AJ}}(\mathbf{F}_4)$ satisfying $\mathrm{H}(\psi) = H$ and $\mathrm{m}(\pi_\psi) = 1$, there are four possible endoscopic types:

(i) $(3, (6,2), (5,1), (3,3))$. A global Arthur parameter of this type is of the form:

$$\mathrm{Sym}^4 \pi_1 \oplus (\mathrm{Sym}^2 \pi_1 \otimes \pi_2[2]) \oplus \mathrm{Sym}^2 \pi_1[3],\ \pi_1, \pi_2 \in \Pi^{\perp}_{\mathrm{alg}}(\mathrm{PGL}_2).$$

(ii) $(3, (9,1), (6,2), (5,1))$. A global Arthur parameter of this type is of the form:

$$\mathrm{Sym}^4 \pi_1 \oplus (\mathrm{Sym}^2 \pi_1 \otimes \pi_2[2]) \oplus (\mathrm{Sym}^2 \pi_1 \otimes \mathrm{Sym}^2 \pi_2),\ \pi_1, \pi_2 \in \Pi^{\perp}_{\mathrm{alg}}(\mathrm{PGL}_2).$$

(iii) $(3, (4,3), (3,3), (1,5))$. A global Arthur parameter of this type is of the form:

$$\mathrm{Sym}^2 \pi_1[3] \oplus (\pi_1 \otimes \pi_2[3]) \oplus [5],\ \pi_1, \pi_2 \in \Pi^{\perp}_{\mathrm{alg}}(\mathrm{PGL}_2).$$

(iv) $(3, (12,1), (9,1), (5,1))$. A global Arthur parameter of this type is of the form:

$$\mathrm{Sym}^4 \pi_1 \oplus (\mathrm{Sym}^2 \pi_1 \otimes \pi_2 \otimes \pi_3) \oplus (\mathrm{Sym}^2 \pi_1 \otimes \mathrm{Sym}^2 \pi_3),\ \pi_1, \pi_2, \pi_3 \in \Pi^{\perp}_{\mathrm{alg}}(\mathrm{PGL}_2).$$

For this subgroup H of F_4, the restriction of the adjoint representation $\mathfrak{f}_4$ of F_4 to H is isomorphic to

$$\begin{gathered}\mathrm{Sym}^4 \mathrm{St} \otimes \left(\mathrm{St} \otimes \mathrm{St} + \mathrm{Sym}^2 \mathrm{St} \otimes \mathbf{1}\right) + \mathrm{Sym}^2 \mathrm{St} \otimes \mathbf{1} \otimes \mathbf{1} \\ + \mathbf{1} \otimes \left(\mathrm{Sym}^2 \mathrm{St} \otimes \mathbf{1} + \mathbf{1} \otimes \mathrm{Sym}^2 \mathrm{St} + \mathrm{Sym}^3 \mathrm{St} \otimes \mathrm{St}\right).\end{gathered}$$

Proposition* 6.3.10. *For a discrete global Arthur parameter $\psi \in \Psi_{\mathrm{AJ}}(\mathbf{F}_4)$ satisfying $\mathrm{H}(\psi) = H$, the multiplicity $\mathrm{m}(\pi_\psi) = 1$ if and only if ψ is one of the following parameters:*

- $\mathrm{Sym}^4 \pi_1 \oplus (\mathrm{Sym}^2 \pi_1 \otimes \pi_2[2]) \oplus \mathrm{Sym}^2 \pi_1[3]$, *where* $\pi_1, \pi_2 \in \Pi^{\perp}_{\mathrm{alg}}(\mathrm{PGL}_2)$ *have motivic weights* w_1, w_2 *respectively such that* $w_2 < 2w_1 - 1$ *or* $w_2 > 4w_1 + 1$.

- $\mathrm{Sym}^4\,\pi_1 \oplus (\mathrm{Sym}^2\,\pi_1 \otimes \pi_2[2]) \oplus (\mathrm{Sym}^2\,\pi_1 \otimes \mathrm{Sym}^2\,\pi_2)$, *where* $\pi_1, \pi_2 \in \Pi_{\mathrm{alg}}^{\perp}(\mathrm{PGL}_2)$ *have motivic weights* w_1, w_2 *respectively and satisfy one of the following conditions:*
  - $2w_1 + 1 < w_2 < 4w_1 - 1,\ w_2 \equiv 1 \bmod 4$*;*
  - $w_2 < 2w_1 - 1$ *or* $w_2 > 4w_1 + 1$*, and* $w_2 \equiv 3 \bmod 4,\ w_1 \neq w_2$*.*
- $\mathrm{Sym}^2\,\pi_1[3] \oplus (\pi_1 \otimes \pi_2[3]) \oplus [5]$, *where* $\pi_1, \pi_2 \in \Pi_{\mathrm{alg}}^{\perp}(\mathrm{PGL}_2)$ *have motivic weights* w_1, w_2 *respectively such that* $w_2 > 3w_1$*.*
- $\mathrm{Sym}^4\,\pi_1 \oplus (\mathrm{Sym}^2\,\pi_1 \otimes \pi_2 \otimes \pi_3) \oplus (\mathrm{Sym}^2\,\pi_1 \otimes \mathrm{Sym}^2\,\pi_3)$, *where* $\pi_1, \pi_2, \pi_3 \in \Pi_{\mathrm{alg}}^{\perp}(\mathrm{PGL}_2)$ *have motivic weights* w_1, w_2, w_3 *respectively such that one of the following conditions holds:*
  - $w_2 > \max(3w_3, 4w_1 + w_3)$*;*
  - $2w_1 + w_3 < w_2 < 4w_1 - w_3$*;*
  - $3w_3 < w_2 < 2w_1 - w_3$*;*
  - $2w_1 + w_3 < w_2 < \min(4w_1 + w_3, 3w_3)$*;*
  - $|4w_1 - w_3| < w_2 < w_3 - 2w_1$*;*
  - $|2w_1 - w_3| < w_2 < \min(4w_1 - w_3, 3w_3)$ *and* $w_3 \neq w_1,\ w_3 \neq w_2$*.*

Proof. We denote the generator of C_ψ by $\gamma = (1, -1, 1) \in \mathrm{Z}(H)$.

Case (i): $\psi = \mathrm{Sym}^4\,\pi_1 \oplus \mathrm{Sym}^2\,\pi_1 \otimes \pi_2[2] \oplus \mathrm{Sym}^2\,\pi_1[3]$, where $\pi_1, \pi_2 \in \Pi_{\mathrm{alg}}^{\perp}(\mathrm{PGL}_2)$ have motivic weights w_1, w_2 respectively. In this case the restriction of $\mathfrak{f}_4$ along ψ is isomorphic to

$$\big(\mathrm{Sym}^4\,\pi_1 \otimes \pi_2[2]\big) \oplus \mathrm{Sym}^4\,\pi_1[3] \oplus \mathrm{Sym}^2\,\pi_1 \oplus \mathrm{Sym}^2\,\pi_2 \oplus \pi_2[4] \oplus [3].$$

By Proposition 5.6.4 we have $\varepsilon_\psi(\gamma) = \varepsilon(\mathrm{Sym}^4\,\pi_1 \otimes \pi_2) \cdot \varepsilon(\pi_2)$. Notice that

$$\varepsilon(\mathbf{I}_w \otimes \mathbf{I}_{w'}) = \varepsilon(\mathbf{I}_{w+w'} + \mathbf{I}_{|w-w'|}) = i^{w+w'+|w-w'|+2} = (-1)^{\max(w,w')+1},$$

thus

$$\varepsilon_\psi(\gamma) = \varepsilon\left((\mathbf{I}_{4w_1} + \mathbf{I}_{3w_1} + \mathbf{I}_{2w_1} + \mathbf{I}_{w_1}) \otimes \mathbf{I}_{w_2}\right) = (-1)^{\max(4w_1,w_2)+\max(2w_1,w_2)}.$$

Hence $\varepsilon_\psi(\gamma) = 1$ if and only if $w_2 < 2w_1$ or $w_2 > 4w_1$. On the other side, $\mu_1 = 2w_1$ or $w_1 + \frac{w_2+1}{2}$. The generator γ of C_ψ acts on $\mathrm{Sym}^4\,\pi_1$ and $\mathrm{Sym}^2\,\pi_1[3]$ by 1 and on $\mathrm{Sym}^2\,\pi_1 \otimes \pi_2[2]$ by -1. We also notice that $\psi \in \Psi_{\mathrm{AJ}}(\mathbf{F}_4)$ implies that $w_2 \notin \{2w_1 \pm 1, 4w_1 \pm 1\}$.

(1) If $w_2 < 2w_1 - 1$, then $\mu_1 = 2w_1$. Now we have $2w_1 > w_1 + \frac{w_2+1}{2} > w_1 + \frac{w_2-1}{2} > w_1 + 1$ and they are larger than other Hodge weights, thus $\mu_4 = w_1 + 1$. Hence $\rho_\psi^\vee(\gamma) = 1$.

(2) If $w_2 > 2w_1 + 1$, then $\mu_1 = w_1 + \frac{w_2+1}{2}$. Now

$$w_1 + \frac{w_2+1}{2} > w_1 + \frac{w_2-1}{2} > \max(2w_1, \frac{w_2+1}{2}) > \min(2w_1, \frac{w_2-1}{2}) \geq w_1 + 1$$

and they are larger than other weights. So $\mu_4 = 2w_1$ or $\frac{w_2+1}{2}, \frac{w_2-1}{2}$. However, if $\mu_4 = 2w_1$, then we must have $\frac{w_2-1}{2} < 2w_1 < \frac{w_2+1}{2}$, which is absurd because there is no integer between $\frac{w_2-1}{2}$ and $\frac{w_2+1}{2}$. Hence $\mu_4 = \frac{w_2 \pm 1}{2}$ and $\rho_\psi^\vee(\gamma) = 1$.

In conclusion, $\rho_\psi^\vee(\gamma) = 1$ for any π_1, π_2. By Arthur's multiplicity formula, $\mathrm{m}(\pi_\psi) = 1$ if and only if $w_2 < 2w_1 - 1$ or $w_2 > 4w_1 + 1$.

Case (ii): $\psi = \mathrm{Sym}^4 \pi_1 \oplus (\mathrm{Sym}^2 \pi_1 \otimes \pi_2[2]) \oplus (\mathrm{Sym}^2 \pi_1 \otimes \mathrm{Sym}^2 \pi_2)$, where $\pi_1, \pi_2 \in \Pi_{\mathrm{alg}}^\perp(\mathrm{PGL}_2)$ have motivic weights w_1, w_2 respectively. In this case the restriction of $\mathfrak{f}_4$ along ψ is isomorphic to

$$\left(\mathrm{Sym}^4 \pi_1 \otimes \mathrm{Sym}^2 \pi_2\right) \oplus \left(\mathrm{Sym}^4 \pi_1 \otimes \pi_2[2]\right) \oplus \mathrm{Sym}^3 \pi_2[2] \oplus \mathrm{Sym}^2 \pi_1 \oplus \mathrm{Sym}^2 \pi_2 \oplus [3].$$

By Proposition 5.6.4 we have:

$$\varepsilon_\psi(\gamma) = \varepsilon(\mathrm{Sym}^4 \pi_1 \otimes \pi_2) \cdot \varepsilon(\mathrm{Sym}^3 \pi_2) = (-1)^{\max(4w_1, w_2) + \max(2w_1, w_2) + (w_2 - 1)/2}.$$

On the other side, γ acts on $\mathrm{Sym}^4 \pi_1, \mathrm{Sym}^2 \pi_1 \otimes \mathrm{Sym}^2 \pi_2$ by 1 and on $\mathrm{Sym}^2 \pi_1 \otimes \pi_2[2]$ by -1.

(1) If $w_1 > w_2$,then $\mu_1 = 2w_1$. Now μ_4 must be $w_1 + \frac{w_2-1}{2}$ and we have $\rho_\psi^\vee(\gamma) = -1$.
(2) If $w_1 < w_2$, then $\mu_1 = w_1 + w_2$. Now $\rho_\psi^\vee(\gamma) = 1$ if and only if μ_4 comes from $\mathrm{Sym}^4 \pi_1$ or $\mathrm{Sym}^2 \pi_1 \otimes \mathrm{Sym}^2 \pi_2$. We can easily verify that none of the weights of these two irreducible summands is possible to be μ_4.

In conclusion, $\rho_\psi^\vee(\gamma) = -1$. By Arthur's multiplicity formula, for $\psi \in \Psi_{\mathrm{AJ}}(\mathbf{F}_4)$ the multiplicity $\mathrm{m}(\pi_\psi) = 1$ if and only if one of the following conditions holds:

- $2w_1 + 1 < w_2 < 4w_1 - 1$, $w_2 \equiv 1 \bmod 4$;
- $w_2 < 2w_1 - 1$ or $w_2 > 4w_1 + 1$, and $w_2 \equiv 3 \bmod 4$, $w_1 \neq w_2$.

Case (iii): $\psi = \mathrm{Sym}^2 \pi_1[3] \oplus (\pi_1 \otimes \pi_2[3]) \oplus [5]$, where $\pi_1, \pi_2 \in \Pi_{\mathrm{alg}}^\perp(\mathrm{PGL}_2)$ have motivic weights w_1, w_2 respectively. In this case, the representations of $\mathrm{SL}_2(\mathbb{C})$ in the restriction of $\mathfrak{f}_4$ along ψ are all odd dimensional, thus $\varepsilon_\psi(\gamma) = 1$ by Proposition 5.6.4. On the other side, γ acts on $\mathrm{Sym}^2 \pi_1[3]$ by 1 and on $\pi_1 \otimes \pi_2[3]$ by -1. We have $\mu_1 = w_1 + 1$ or $\frac{w_1+w_2}{2} + 1$.

(1) If $w_1 > w_2$, then $\mu_1 = w_1 + 1$. The condition that $\psi \in \Psi_{\mathrm{AJ}}(\mathbf{F}_4)$ implies that $w_1 > w_2 + 4$, thus $w_1 + 1 > w_1 > w_1 - 1 > \frac{w_1+w_2}{2} + 1$, which are larger than other weights. So $\mu_4 = \frac{w_1+w_2}{2} + 1$ and $\rho_\psi^\vee(\gamma) = -1$.
(2) If $w_1 < w_2$, then $\mu_1 = \frac{w_1+w_2}{2} + 1$. Similarly, we have $w_1 < w_2 - 4$. Now μ_4 must be $w_1 + 1$ or $\frac{w_2-w_1}{2} + 1$, so $\rho_\psi^\vee(\gamma) = 1$ if and only if $\mu_4 = \frac{w_2-w_1}{2} + 1$. This is equivalent to $w_2 > 3w_1$.

By Arthur's multiplicity formula, $\mathrm{m}(\pi_\psi) = 1$ if and only if $w_2 > 3w_1$.

Case (iv): $\psi = \mathrm{Sym}^4 \pi_1 \oplus (\mathrm{Sym}^2 \pi_1 \otimes \pi_2 \otimes \pi_3) \oplus (\mathrm{Sym}^2 \pi_1 \otimes \mathrm{Sym}^2 \pi_3)$, where $\pi_1, \pi_2, \pi_3 \in \Pi_{\mathrm{alg}}^\perp(\mathrm{PGL}_2)$ have motivic weights w_1, w_2, w_3 respectively. In this case, $\varepsilon_\psi(\gamma) = 1$ since the parameter is tempered. On the other side, γ acts on $\mathrm{Sym}^4 \pi_1$ and $\mathrm{Sym}^2 \pi_1 \otimes \mathrm{Sym}^2 \pi_3$ by 1 and on $\mathrm{Sym}^2 \pi_1 \otimes \pi_2 \otimes \pi_3$ by -1. We denote the ratios $w_1/w_3, w_2/w_3$ by r_1, r_2 respectively, and denote the multiset of elements μ/w_3, μ running over the eigenvalues of $\mathrm{c}_\infty(\psi)$, by $\widetilde{\mathrm{W}}$. We still order the elements of $\widetilde{\mathrm{W}}$ by $\mu_1 > \mu_2 > \cdots > \mu_{26}$. The largest one μ_1 must be $r_1 + 1$ or $2r_1$ or $r_1 + \frac{r_2+1}{2}$.

(1) If $r_1 < 1, r_2 < 1$, then $\mu_1 = r_1 + 1$. Now $\mu_2 = 2r_1$ or 1 or $r_1 + \frac{r_2+1}{2}$.
    (a) If $r_1 > 1/2$ and $r_2 < 2r_1 - 1$, then $\mu_2 = 2r_1$. Now $r_1 + 1 > 2r_1 > r_1 + \frac{r_2+1}{2} > r_1 + \frac{1-r_2}{2}$, which are larger than other 22 elements, thus $\mu_4 = r_1 + \frac{1-r_2}{2}$ and $\rho_\psi^\vee(\gamma) = -1$.

(b) If $r_1 < 1/2$ and $r_2 < 1-2r_1$, then $\mu_2 = 1$. Now $\rho_\psi^\vee(\gamma) = 1$ if and only if $\mu_4 = 1-r_1$, which is equivalent to $|4r_1 - 1| < r_2$.

(c) If $r_2 > |2r_1 - 1|$, then $\mu_2 = r_1 + \frac{r_2+1}{2}$. Now $\rho_\psi^\vee(\gamma) = 1$ if and only if $\mu_4 = 2r_1$ or 1, which is equivalent to $r_2 < 4r_1 - 1$.

(2) If $r_1 > 1, r_2 < 2r_1 - 1$, then $\mu_1 = 2r_1$. Now $\rho_\psi^\vee(\gamma) = 1$ if and only if $\mu_4 = r_1 + 1$, which is equivalent to $r_2 > 3$.

(3) If $r_2 > 1, r_2 > 2r_1 - 1$, then $\mu_1 = r_1 + \frac{r_2+1}{2}$. Now μ_2 belongs to the (multi)set $\{r_1 + 1, 2r_1, r_1 + \frac{r_2-1}{2}, \frac{r_2+1}{2}\}$.

(a) If $r_1 < 1$ and $r_2 < 2r_1 + 1$, then $\mu_2 = r_1 + 1$. Now $\rho_\psi^\vee(\gamma) = 1$ if and only if $\mu_4 = \frac{r_2+1}{2}$, which is equivalent to $r_2 < 4r_1 - 1$.

(b) If $r_1 > 1$ and $r_2 < 2r_1 + 1$, then $\mu_2 = 2r_1$. Now $\mu_4 = \min(r_1 + 1, r_1 + \frac{r_2-1}{2})$, thus $\rho_\psi^\vee(\gamma) = 1$ if and only if $r_2 < 3$.

(c) If $r_1 > 1$ and $r_2 > 2r_1 + 1$, then $\mu_2 = r_1 + \frac{r_2-1}{2}$. Now $\rho_\psi^\vee(\gamma) = 1$ if and only if $\mu_4 = \frac{r_2\pm1}{2}$, which is equivalent to $r_2 < 4r_1 - 1$ or $r_2 > 4r_1 + 1$.

(d) If $r_1 < 1$ and $r_2 > 2r_1 + 1$, then $\mu_2 = \frac{r_2+1}{2}$. Now $\rho_\psi^\vee(\gamma) = 1$ if and only if $\mu_4 = r_1 + \frac{r_2-1}{2}$ or $\frac{r_2+1}{2} - r_1$, which is equivalent to that $r_2 < \min(3, 4r_1 + 1)$ or $r_2 > \max(3, 4r_1 + 1)$.

In conclusion, by Arthur's multiplicity formula, $\mathrm{m}(\pi_\psi) = 1$ if and only if w_1, w_2, w_3 satisfy one of the conditions listed in the proposition. $\square$

6.3.7 $H = \left(\mathrm{A}_1^{[5,4^4,1^5]} \times \mathrm{A}_1^{[2^6,1^{14}]} \times \mathrm{A}_1^{[2^6,1^{14}]}\right)/\mu_2^\Delta$

By §4.6.9, the restriction of the 26-dimensional irreducible representation J_0 of F_4 to H is isomorphic to

$$\mathbf{1} + \mathbf{1} \otimes \mathrm{St} \otimes \mathrm{St} + \mathrm{Sym}^3 \mathrm{St} \otimes (\mathrm{St} \otimes \mathbf{1} + \mathbf{1} \otimes \mathrm{St}) + \mathrm{Sym}^4 \mathrm{St} \otimes \mathbf{1} \otimes \mathbf{1},$$

and the centralizer of H in F_4 is $\mathrm{Z}(H) \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

For $\psi \in \Psi_{\mathrm{AJ}}(\mathbf{F}_4)$ satisfying $\mathrm{H}(\psi) = H$ and $\mathrm{m}(\pi_\psi) = 1$, there are three possible endoscopic types:

(i) $(5, (8,1), (5,1), (4,2), (2,2), (1,1))$. A global Arthur parameter of this type is of the form:

$$\mathrm{Sym}^4 \pi_1 \oplus (\mathrm{Sym}^3 \pi_1 \otimes \pi_2) \oplus \mathrm{Sym}^3 \pi_1[2] \oplus \pi_2[2] \oplus [1],\ \pi_1, \pi_2 \in \Pi_{\mathrm{alg}}^\perp(\mathrm{PGL}_2).$$

(ii) $(5, (4,1), (2,4), (2,4), (1,5), (1,1))$. A global Arthur parameter of this type is of the form:

$$(\pi_1 \otimes \pi_2) \oplus \pi_1[4] \oplus \pi_2[4] \oplus [5] \oplus [1],\ \pi_1, \pi_2 \in \Pi_{\mathrm{alg}}^\perp(\mathrm{PGL}_2).$$

(iii) $(5, (8,1), (8,1), (5,1), (4,1), (1,1))$. A global Arthur parameter of this type is of the form:

$$\mathrm{Sym}^4 \pi_1 \oplus (\mathrm{Sym}^3 \pi_1 \otimes \pi_2) \oplus (\mathrm{Sym}^3 \pi_1 \otimes \pi_3) \oplus (\pi_2 \otimes \pi_3) \oplus [1],\ \pi_1, \pi_2, \pi_3 \in \Pi_{\mathrm{alg}}^\perp(\mathrm{PGL}_2).$$

For this subgroup H of F_4, the restriction of the adjoint representation $\mathfrak{f}_4$ of F_4 to H is isomorphic to

$$\mathbf{1}\otimes\left(\mathrm{Sym}^2\,\mathrm{St}\otimes\mathbf{1}+\mathbf{1}\otimes\mathrm{Sym}^2\,\mathrm{St}\right)+\mathrm{Sym}^2\,\mathrm{St}\otimes\mathbf{1}\otimes\mathbf{1}+\mathrm{Sym}^3\,\mathrm{St}\otimes(\mathrm{St}\otimes\mathbf{1}+\mathbf{1}\otimes\mathrm{St})$$
$$+\mathrm{Sym}^4\,\mathrm{St}\otimes\mathrm{St}\otimes\mathrm{St}+\mathrm{Sym}^6\,\mathrm{St}\otimes\mathbf{1}\otimes\mathbf{1}$$

Proposition* 6.3.11. *For a discrete global Arthur parameter $\psi\in\Psi_{\mathrm{AJ}}(\mathbf{F}_4)$ satisfying $\mathrm{H}(\psi)=H$, the multiplicity $\mathrm{m}(\pi_\psi)=1$ if and only if ψ is one of the following parameters:*

- $\mathrm{Sym}^4\,\pi_1\oplus(\mathrm{Sym}^3\,\pi_1\otimes\pi_2)\oplus\mathrm{Sym}^3\,\pi_1[2]\oplus\pi_2[2]\oplus[1]$, *where $\pi_1,\pi_2\in\Pi_{\mathrm{alg}}^{\perp}(\mathrm{PGL}_2)$ have motivic weights w_1,w_2 respectively and satisfy one of the following conditions*
  - $w_2<w_1$ *or* $w_2>4w_1+1$, *and* $w_2\equiv 3\bmod 4$*;*
  - $3w_1<w_2<4w_1-1$ *and* $w_2\equiv 1\bmod 4$.
- $(\pi_1\otimes\pi_2)\oplus\pi_1[4]\oplus\pi_2[4]\oplus[5]\oplus[1]$, *where $\pi_1,\pi_2\in\Pi_{\mathrm{alg}}^{\perp}(\mathrm{PGL}_2)$ have motivic weights $w_1>w_2$ respectively and $w_1\equiv 3\bmod 4$, $w_2\equiv 1\bmod 4$, $w_2<w_1-4$.*
- $\mathrm{Sym}^4\,\pi_1\oplus(\mathrm{Sym}^3\,\pi_1\otimes\pi_2)\oplus(\mathrm{Sym}^3\,\pi_1\otimes\pi_3)\oplus(\pi_2\otimes\pi_3)\oplus[1]$, *where $\pi_1,\pi_2,\pi_3\in\Pi_{\mathrm{alg}}^{\perp}(\mathrm{PGL}_2)$ have motivic weights w_1 and $w_2>w_3$ respectively satisfying one of the following conditions:*
  - $w_1>w_3$ *and* $2w_1-w_3<w_2<2w_1+w_3$*;*
  - $w_3<3w_1<w_2<2w_1+w_3$*;*
  - $w_1<w_3<3w_1$, $w_2>4w_1+w_3$.

Proof. We take a set of generators $\{\sigma=(-1,1,1),\sigma_1=(1,1,-1)\}$ of $\mathrm{C}_\psi=\mathrm{Z}(H)\simeq\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$. Let χ_1,χ_2 be two generators of the character group of C_ψ such that $\chi_1(\sigma)=\chi_2(\sigma_1)=-1,\chi_1(\sigma_1)=\chi_2(\sigma)=1$.

Case (i): $\psi=\mathrm{Sym}^4\,\pi_1\oplus(\mathrm{Sym}^3\,\pi_1\otimes\pi_2)\oplus\mathrm{Sym}^3\,\pi_1[2]\oplus\pi_2[2]\oplus[1]$, where $\pi_1,\pi_2\in\Pi_{\mathrm{alg}}^{\perp}(\mathrm{PGL}_2)$ have motivic weights w_1,w_2 respectively. In this case, the restriction of $\mathfrak{f}_4$ along ψ is isomorphic to:

$$\mathrm{Sym}^6\,\pi_1\oplus\left(\mathrm{Sym}^4\,\pi_1\otimes\pi_2[2]\right)\oplus\left(\mathrm{Sym}^3\,\pi_1\otimes\pi_2\right)\oplus\mathrm{Sym}^3\,\pi_1[2]\oplus\mathrm{Sym}^2\,\pi_1\oplus\mathrm{Sym}^2\,\pi_2\oplus[3].$$

By Proposition 5.6.4 we have:

$$\varepsilon_\psi(\sigma)=\varepsilon(\mathrm{Sym}^3\,\pi_1)=\varepsilon(\mathbf{I}_{3w_1}+\mathbf{I}_{w_1})=(-1)^{(3w_1+1)/2+(w_1+1)/2}=-1,$$
$$\varepsilon_\psi(\sigma_1)=\varepsilon(\mathrm{Sym}^4\,\pi_1\otimes\pi_2)\cdot\varepsilon(\mathrm{Sym}^3\,\pi_1)=(-1)^{\max(4w_1,w_2)+\max(2w_1,w_2)+(w_2-1)/2}.$$

So $\varepsilon_\psi=\chi_1$ or $\chi_1\chi_2$. On the other side, the largest weight μ_1 is $2w_1$ or $\frac{3w_1+w_2}{2}$.

(1) If $w_1>w_2$, then $\mu_1=2w_1$. Now $2w_1>\frac{3w_1+w_2}{2}>\frac{3w_1+1}{2}>\frac{3w_1-1}{2}$ and they are larger than other weights, thus $\mu_4=\frac{3w_1-1}{2}$ and $\rho_\psi^\vee=\chi_1\chi_2$.

(2) If $w_1<w_2$, then $\mu_1=\frac{3w_1+w_2}{2}$. Now $\mu_2=2w_1$ or $\frac{w_1+w_2}{2}$.

  (a) If $w_2<3w_1$, then $\mu_2=2w_1$. Now $\mu_4=\frac{w_1+w_2}{2}$ or $\frac{3w_1\pm1}{2}$, thus $\rho_\psi^\vee=1$ or χ_2.

  (b) If $w_2>3w_1$, then $\mu_2=\frac{w_1+w_2}{2}$. Now $\mu_4=2w_1$ or $\frac{w_2\pm1}{2}$, thus $\rho^\vee=\chi_1$ or $\chi_1\chi_2$. Notice that $\mu_4=2w_1$ if and only if $2w_1$ lies between $\frac{w_2+1}{2}$ and $\frac{w_2-1}{2}$, which can not happen. So $\rho_\psi^\vee=\chi_1\chi_2$ for any $w_2>3w_1$ and $w_2\neq 4w_1\pm1$.

Hence by Arthur's multiplicity formula, $\mathrm{m}(\pi_\psi) = 1$ if and only if one of the following conditions holds:

- $w_2 < w_1$ or $w_2 > 4w_1 + 1$, and $w_2 \equiv 3 \bmod 4$;
- $3w_1 < w_2 < 4w_1 - 1$, and $w_2 \equiv 1 \bmod 4$.

Case (ii): $\psi = (\pi_1 \otimes \pi_2) \oplus \pi_1[4] \oplus \pi_2[4] \oplus [5] \oplus [1]$, where $\pi_1, \pi_2 \in \Pi_{\mathrm{alg}}^{\perp}(\mathrm{PGL}_2)$ have motivic weights $w_1 > w_2$ respectively. In this case, the restriction of $\mathfrak{f}_4$ along ψ is isomorphic to

$$\mathrm{Sym}^2 \pi_1 \oplus \mathrm{Sym}^2 \pi_2 \oplus (\pi_1 \otimes \pi_2[5]) \oplus \pi_1[4] \oplus \pi_2[4] \oplus [7] \oplus [3].$$

By Proposition 5.6.4 we have:

$$\varepsilon_\psi(\sigma) = \varepsilon(\pi_1) \cdot \varepsilon(\pi_2) = \varepsilon(\mathbf{I}_{w_1}) \cdot \varepsilon(\mathbf{I}_{w_2}) = (-1)^{(w_1+w_2)/2+1}$$
$$\varepsilon_\psi(\sigma_1) = \varepsilon(\pi_2) = \varepsilon(\mathbf{I}_{w_2}) = (-1)^{(w_2+1)/2}.$$

On the other side, the condition $\psi \in \Psi_{\mathrm{AJ}}(\mathbf{F}_4)$ implies that $w_2 < w_1 - 4$. Since

$$\frac{w_1 + w_2}{2} > \frac{w_1 + 3}{2} > \frac{w_1 + 1}{2} > \frac{w_1 - 1}{2}$$

and they are larger than other weights, we have $\mu_1 = \frac{w_1+w_2}{2}$ and $\mu_4 = \frac{w_1-1}{2}$. The global component group C_ψ acts on $\pi_1 \otimes \pi_2$ and $\pi_1[4]$ by χ_2 and χ_1 respectively, thus by Proposition 6.2.1 the character $\rho_\psi^\vee = \chi_1\chi_2$. By Arthur's multiplicity formula, $\mathrm{m}(\pi_\psi) = 1$ if and only if $w_1 \equiv 3 \bmod 4, w_2 \equiv 1 \bmod 4$ and $w_2 < w_1 - 4$.

Case (iii): $\psi = \mathrm{Sym}^4 \pi_1 \oplus (\mathrm{Sym}^3 \pi_1 \otimes \pi_2) \oplus (\mathrm{Sym}^3 \pi_1 \otimes \pi_3) \oplus (\pi_2 \otimes \pi_3) \oplus [1]$, where $\pi_1, \pi_2, \pi_3 \in \Pi_{\mathrm{alg}}^{\perp}(\mathrm{PGL}_2)$ have motivic weights w_1, w_2, w_3 respectively and we assume that $w_2 > w_3$. In this case ε_ψ is trivial since ψ is tempered. On the other side, C_ψ acts on the four summands $\mathrm{Sym}^4 \pi_1, \mathrm{Sym}^3 \pi_1 \otimes \pi_2, \mathrm{Sym}^3 \pi_1 \otimes \pi_3$ and $\pi_2 \otimes \pi_3$ by $1, \chi_1, \chi_1\chi_2$ and χ_2 respectively. Denote the ratios $w_1/w_3, w_2/w_3$ by r_1, r_2 respectively and the corresponding multiset by $\widetilde{\mathcal{W}}$ as in the proof of Proposition 6.3.10. We still order the elements of $\widetilde{\mathcal{W}}$ by $\mu_1 > \mu_2 > \cdots > \mu_{26}$, then by Proposition 6.2.1 the character $\rho_\psi^\vee = 1$ if and only if μ_1 and μ_4 come from the same irreducible summand of ψ. The largest element μ_1 is $2r_1$ or $\frac{3r_1+r_2}{2}$ or $\frac{r_2+1}{2}$.

(1) If $r_2 < r_1$, then $\mu_1 = 2r_1$. Now $2r_1 > \frac{3r_1+r_2}{2} > \frac{3r_1+1}{2} > \frac{3r_1-r_2}{2} > r_1$, thus $\rho_\psi^\vee$ is not trivial.
(2) If $r_2 > r_1$ and $r_1 > 1/3$, then $\mu_1 = \frac{3r_1+r_2}{2}$.
   (a) If $r_1 > 1$, then $\rho_\psi^\vee = 1$ if and only if $\mu_4 = \frac{r_1+r_2}{2}$, which is equivalent to $2r_1 - 1 < r_2 < 2r_1 + 1$.
   (b) If $r_1 < 1$, then $\rho_\psi^\vee = 1$ if and only if $\mu_4 = \frac{r_2 \pm r_1}{2}$.
      (I) $\mu_4 = \frac{r_2+r_1}{2}$ if and only if $2r_1 < \frac{r_2+r_1}{2} < \frac{3r_1+1}{2} \Leftrightarrow 3r_1 < r_2 < 2r_1 + 1$.
      (II) $\mu_4 = \frac{r_2-r_1}{2}$ if and only if $\frac{r_2-r_1}{2} > \frac{3r_1+1}{2} \Leftrightarrow r_2 > 4r_1 + 1$.
(3) If $r_1 < 1/3$, then $\mu_1 = \frac{r_2+1}{2}$. Now $\frac{r_2+1}{2}, \frac{r_2 \pm r_1}{2}, \frac{3r_1+r_2}{2}$ are larger than $\frac{r_2-1}{2}$, so $\frac{r_2-1}{2}$ can not be μ_4 and thus $\rho_\psi^\vee \neq 1$.

In conclusion, by Arthur's multiplicity formula, $\mathrm{m}(\pi_\psi) = 1$ if and only if w_1, w_2, w_3 satisfy one of the three conditions in Proposition 6.3.11. $\square$

6.3.8 $H = \prod_{i=1}^4 \mathrm{A}_1^{[2^6,1^{14}]}/\mu_2^\Delta$

By §4.6.13, the restriction of the 26-dimensional irreducible representation J_0 of F_4 to H is isomorphic to

$$\mathbf{1}^{\oplus 2} + \sum_{\mathrm{Sym}} \mathrm{St} \otimes \mathrm{St} \otimes \mathbf{1} \otimes \mathbf{1},$$

and the centralizer of H in F_4 is $\mathrm{Z}(H) \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

For $\psi \in \Psi_{\mathrm{AJ}}(\mathbf{F}_4)$ satisfying $\mathrm{H}(\psi) = H$ and $\mathrm{m}(\pi_\psi) = 1$, there are two possible endoscopic types:

(i) $(8, (4, 1), (4, 1), (4, 1), (2, 2), (2, 2), (2, 2), (1, 1), (1, 1))$. A global Arthur parameter of this type is of the form:

$$\left(\bigoplus_{1\leq i<j\leq 3} \pi_i \otimes \pi_j\right) \oplus \left(\bigoplus_{1\leq i\leq 3} \pi_i[2]\right) \oplus [1] \oplus [1],\ \pi_1, \pi_2, \pi_3 \in \Pi_{\mathrm{alg}}^{\perp}(\mathrm{PGL}_2).$$

(ii) $(8, (4, 1), (4, 1), (4, 1), (4, 1), (4, 1), (4, 1), (1, 1), (1, 1))$. A global Arthur parameter of this type is of the form:

$$\left(\bigoplus_{1\leq i<j\leq 4} \pi_i \otimes \pi_j\right) \oplus [1] \oplus [1],\ \pi_1, \pi_2, \pi_3, \pi_4 \in \Pi_{\mathrm{alg}}^{\perp}(\mathrm{PGL}_2).$$

For this subgroup H of F_4, the restriction of the adjoint representation $\mathfrak{f}_4$ of F_4 to H is isomorphic to

$$\sum_{\mathrm{Sym}} \mathrm{Sym}^2 \mathrm{St} \otimes \mathbf{1} \otimes \mathbf{1} \otimes \mathbf{1} + \sum_{\mathrm{Sym}} \mathrm{St} \otimes \mathrm{St} \otimes \mathbf{1} \otimes \mathbf{1} + \mathrm{St} \otimes \mathrm{St} \otimes \mathrm{St} \otimes \mathrm{St}.$$

Proposition* 6.3.12. *For a discrete global Arthur parameter $\psi \in \Psi_{\mathrm{AJ}}(\mathbf{F}_4)$ satisfying $\mathrm{H}(\psi) = H$, the multiplicity $\mathrm{m}(\pi_\psi) = 1$ if and only if ψ has the form:*

$$\psi = \left(\bigoplus_{1\leq i<j\leq 3} \pi_i \otimes \pi_j\right) \oplus \left(\bigoplus_{1\leq i\leq 3} \pi_i[2]\right) \oplus [1] \oplus [1],$$

where $\pi_1, \pi_2, \pi_3 \in \Pi_{\mathrm{alg}}^{\perp}(\mathrm{PGL}_2)$ have motivic weights $w_1 > w_2 > w_3$ respectively such that one of the following conditions holds:

- $w_1 > w_2 + w_3 + 1$, *and* $w_1 \equiv w_3 \equiv 3 \bmod 4$, $w_2 \equiv 1 \bmod 4$;
- $w_1 < w_2 + w_3 - 1$, *and* $w_1 \equiv w_3 \equiv 1 \bmod 4$, $w_2 \equiv 3 \bmod 4$.

Proof. We take a set of generators $\{\gamma = (-1, 1, 1, 1), \gamma_1 = (1, -1, 1, 1), \gamma_2 = (1, 1, -1, 1)\}$ of $\mathrm{C}_\psi = \mathrm{Z}(H) \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

Case (i): $\psi = (\bigoplus_{1\leq i<j\leq 3} \pi_i \otimes \pi_j) \oplus (\bigoplus_{1\leq i\leq 3} \pi_i[2]) \oplus [1] \oplus [1]$, where $\pi_1, \pi_2, \pi_3 \in \Pi_{\mathrm{alg}}^{\perp}(\mathrm{PGL}_2)$ have motivic weights $w_1 > w_2 > w_3$ respectively. In this case, the restriction of $\mathfrak{f}_4$ along ψ is isomorphic to

$$(\pi_1 \otimes \pi_2 \otimes \pi_3[2]) \oplus \left(\bigoplus_{1\leq i<j\leq 3} \pi_i \otimes \pi_j\right) \oplus \left(\bigoplus_{1\leq i\leq 3} \mathrm{Sym}^2 \pi_i\right) \oplus \left(\bigoplus_{1\leq i\leq 3} \pi_i[2]\right) \oplus [3].$$

By Proposition 5.6.4 we have:

$$\varepsilon_\psi(\gamma) = \varepsilon(\pi_1)\cdot\varepsilon(\pi_1\otimes\pi_2\otimes\pi_3) = (-1)^{\max(w_1,w_2+w_3)+(w_1-1)/2},$$
$$\varepsilon_\psi(\gamma_1) = \varepsilon(\pi_2)\cdot\varepsilon(\pi_1\otimes\pi_2\otimes\pi_3) = (-1)^{\max(w_1,w_2+w_3)+(w_2-1)/2},$$
$$\varepsilon_\psi(\gamma_2) = \varepsilon(\pi_3)\cdot\varepsilon(\pi_1\otimes\pi_2\otimes\pi_3) = (-1)^{\max(w_1,w_2+w_3)+(w_3-1)/2}.$$

On the other side, the largest element μ_1 must be $\frac{w_1+w_2}{2}$ and μ_4 is the middle one of $\{\frac{w_1+1}{2},\frac{w_1-1}{2},\frac{w_2+w_3}{2}\}$. Since there is no integer between $\frac{w_1+1}{2}$ and $\frac{w_1-1}{2}$, we have $\mu_4\neq\frac{w_2+w_3}{2}$. So $\rho_\psi^\vee$ is the product of two characters of C_ψ coming from $\pi_1\otimes\pi_2$ and $\pi_1[2]$ respectively, thus $\rho_\psi^\vee(\gamma)=\rho_\psi^\vee(\gamma_2)=1$ and $\rho_\psi^\vee(\gamma_1)=-1$.

By Arthur's multiplicity formula, $\mathrm{m}(\pi_\psi)=1$ if and only if one of the following conditions holds:

- $w_1>w_2+w_3+1$, and $w_1\equiv w_3\equiv 3 \bmod 4, w_2\equiv 1\bmod 4$;
- $w_1<w_2+w_3-1$, and $w_1\equiv w_3\equiv 1 \bmod 4, w_2\equiv 3\bmod 4$.

Case (ii): $\psi=(\bigoplus_{1\leq i<j\leq 4}\pi_i\otimes\pi_j)\oplus[1]\oplus[1]$, where $\pi_1,\pi_2,\pi_3,\pi_4\in\Pi_{\mathrm{alg}}^{\perp}(\mathrm{PGL}_2)$ have motivic weights $w_1>w_2>w_3>w_4$ respectively. In this case, ε_ψ is trivial. On the other side, μ_1 must be $\frac{w_1+w_2}{2}$. Notice that C_ψ acts on 6 components $\pi_i\otimes\pi_j$ via 6 different characters, so $\rho_\psi^\vee$ is trivial if and only if $\mu_4=\frac{w_1-w_2}{2}$. However,

$$\frac{w_1-w_2}{2}<\frac{w_1-w_3}{2}<\frac{w_1-w_4}{2}<\frac{w_1+w_4}{2}<\frac{w_1+w_3}{2}<\frac{w_1+w_2}{2},$$

thus $\rho_\psi^\vee\neq 1$ and $\mathrm{m}(\pi_\psi)=0$. □

6.3.9 $H=\mathrm{A}_1^{[5,3^7]}\times\mathrm{G}_2$

In this case, we need to consider cuspidal representations $\pi\in\Pi_{\mathrm{alg,reg}}^{\mathrm{o}}(\mathrm{PGL}_7)$ such that the image of the corresponding irreducible representation $\mathcal{L}_\mathbb{Z}\to\mathrm{SL}_7(\mathbb{C})$ is a compact Lie group of type G_2. This kind of representations correspond to discrete automorphic representations of the unique semisimple anisotropic $\mathbb{Z}$-group of type G_2 with stable tempered type, which have been studied in [CR15, §8], conditional to the existence of $\mathcal{L}_\mathbb{Z}$ and Arthur's multiplicity formula. We denote by $\Pi_{\mathrm{alg}}^{\mathrm{G}_2}(\mathrm{PGL}_7)\subset\Pi_{\mathrm{alg,reg}}^{\mathrm{o}}(\mathrm{PGL}_7)$ the subset of these representations. The Hodge weights of a representation $\pi\in\Pi_{\mathrm{alg}}^{\mathrm{G}_2}(\mathrm{PGL}_7)$ have the form $w+v>w>v$, where w,v are even integers.

By §4.6.4, the restriction of the 26-dimensional irreducible representation J_0 of F_4 to H is isomorphic to

$$\mathrm{Sym}^2\,\mathrm{St}\otimes\mathrm{V}_7+\mathrm{Sym}^4\,\mathrm{St}\otimes\mathbf{1},$$

where V_7 is the 7-dimensional irreducible representation of G_2, and the centralizer of H in F_4 is trivial.

For $\psi\in\Psi_{\mathrm{AJ}}(\mathbf{F}_4)$ satisfying $\mathrm{H}(\psi)=H$ and $\mathrm{m}(\pi_\psi)=1$, there are two possible endoscopic types:

(i) $(2,(7,3),(1,5))$. A global Arthur parameter of this type is of the form:

$$\pi[3]\oplus[5],\ \pi\in\Pi_{\mathrm{alg}}^{\mathrm{G}_2}(\mathrm{PGL}_7).$$

(ii) $(2,(21,1),(5,1))$. A global Arthur parameter of this type is of the form:

$$(\pi \otimes \mathrm{Sym}^2 \tau) \oplus \mathrm{Sym}^4 \tau,\ \pi \in \Pi_{\mathrm{alg}}^{\mathrm{G}_2}(\mathrm{PGL}_7), \tau \in \Pi_{\mathrm{alg}}^{\perp}(\mathrm{PGL}_2).$$

Proposition* 6.3.13. *For a discrete global Arthur parameter $\psi \in \Psi_{\mathrm{AJ}}(\mathbf{F}_4)$ satisfying $\mathrm{H}(\psi) = H$, the multiplicity $\mathrm{m}(\pi_\psi) = 1$ if and only if ψ is one of the following parameters:*

- $\pi[3] \oplus [5]$, *where $\pi \in \Pi_{\mathrm{alg}}^{\mathrm{G}_2}(\mathrm{PGL}_7)$ has Hodge weights $w + v > w > v$ such that $v > 4$;*
- $(\pi \otimes \mathrm{Sym}^2 \tau) \oplus \mathrm{Sym}^4 \tau$, *where $\pi \in \Pi_{\mathrm{alg}}^{\mathrm{G}_2}(\mathrm{PGL}_7)$ has Hodge weights $w + v > w > v$ and $\tau \in \Pi_{\mathrm{alg}}^{\perp}(\mathrm{PGL}_2)$ satisfies $\mathrm{w}(\tau) \notin \{\frac{w+v}{2}, \frac{w}{2}, \frac{v}{2}\}$.*

Proof. This follows from the condition $\psi \in \Psi_{\mathrm{AJ}}(\mathbf{F}_4)$ and the fact that C_ψ is trivial. $\square$

6.3.10 $H = \left(\mathrm{A}_1^{[2^6,1^{14}]} \times \mathrm{A}_1^{[2^6,1^{14}]} \times \mathrm{Sp}(2)\right)/\mu_2^{\Delta}$

By §4.6.6, the restriction of the 26-dimensional irreducible representation J_0 of F_4 to H is isomorphic to

$$\mathbf{1} + \mathrm{St} \otimes \mathrm{St} \otimes \mathbf{1} + \mathrm{St} \otimes \mathbf{1} \otimes \mathrm{V}_4 + \mathbf{1} \otimes \mathrm{St} \otimes \mathrm{V}_4 + \mathbf{1} \otimes \mathbf{1} \otimes \wedge^* \mathrm{V}_4,$$

where V_4 is the standard representation of $\mathrm{Sp}(2)$ and $\wedge^*\mathrm{V}_4$ is the 5-dimensional irreducible representation of $\mathrm{Sp}(2)$. The centralizer of H in F_4 is $\mathrm{Z}(H) \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

For any $\pi \in \Pi_{\mathrm{alg}}^{\mathrm{Sp}_4}(\mathrm{PGL}_4)$, we denote by $\wedge^*\pi$ the representation in $\Pi_{\mathrm{alg,reg}}^{\mathrm{o}}(\mathrm{PGL}_5)$ corresponding to the following irreducible representation of $\mathcal{L}_{\mathbb{Z}}$:

$$\mathcal{L}_{\mathbb{Z}} \xrightarrow{\psi_\pi} \mathrm{Sp}(2) \xrightarrow{\wedge^*} \mathrm{SL}_5(\mathbb{C}).$$

For $\psi \in \Psi_{\mathrm{AJ}}(\mathbf{F}_4)$ satisfying $\mathrm{H}(\psi) = H$ and $\mathrm{m}(\pi_\psi) = 1$, there are two possible endoscopic types:

(i) $(5,(8,1),(5,1),(4,2),(2,2),(1,1))$. A global Arthur parameter of this type is of the form:

$$\wedge^*\pi \oplus (\pi \otimes \tau) \oplus \pi[2] \oplus \tau[2] \oplus [1],\ \pi \in \Pi_{\mathrm{alg}}^{\mathrm{Sp}_4}(\mathrm{PGL}_4), \tau \in \Pi_{\mathrm{alg}}^{\perp}(\mathrm{PGL}_2).$$

(ii) $(5,(8,1),(8,1),(5,1),(4,1),(1,1))$. A global Arthur parameter of this type is of the form:

$$\wedge^*\pi \oplus (\pi \otimes \tau_1) \oplus (\pi \otimes \tau_2) \oplus (\tau_1 \otimes \tau_2) \oplus [1],\ \pi \in \Pi_{\mathrm{alg}}^{\mathrm{Sp}_4}(\mathrm{PGL}_4), \tau_1, \tau_2 \in \Pi_{\mathrm{alg}}^{\perp}(\mathrm{PGL}_2).$$

For this subgroup H of F_4, the restriction of the adjoint representation $\mathfrak{f}_4$ of F_4 to H is isomorphic to

$$\left(\mathrm{Sym}^2 \mathrm{St} \otimes \mathbf{1} + \mathbf{1} \otimes \mathrm{Sym}^2 \mathrm{St}\right) \otimes \mathbf{1} + (\mathrm{St} \otimes \mathbf{1} + \mathbf{1} \otimes \mathrm{St}) \otimes \mathrm{V}_4 \\ + \mathrm{St} \otimes \mathrm{St} \otimes \wedge^* \mathrm{V}_4 + \mathbf{1} \otimes \mathbf{1} \otimes \mathrm{Sym}^2 \mathrm{V}_4.$$

Proposition* 6.3.14. *For a discrete global Arthur parameter $\psi \in \Psi_{\mathrm{AJ}}(\mathbf{F}_4)$ satisfying $\mathrm{H}(\psi) = H$, the multiplicity $\mathrm{m}(\pi_\psi) = 1$ if and only if ψ is one of the following parameters:*

- $\wedge^*\pi \oplus (\pi\otimes\tau)\oplus\pi[2]\oplus\tau[2]\oplus[1]$, *where* $\pi \in \Pi_{\mathrm{alg}}^{\mathrm{Sp}_4}(\mathrm{PGL}_4)$ *has Hodge weights* $w_1 > w_2 > 1$ *and* $\tau \in \Pi_{\mathrm{alg}}^{\perp}(\mathrm{PGL}_2)$ *has motivic weight* v *satisfying one of the following conditions:*
  - $w_1 < v < w_1+w_2-1, w_1+w_2 \equiv 0 \bmod 4, v \equiv 1 \bmod 4$;
  - $w_1-w_2+1 < v < w_2, w_1+w_2 \equiv 0 \bmod 4, v \equiv 1 \bmod 4$;
  - $w_2 < v < w_1-w_2-1, w_1+w_2 \equiv 2 \bmod 4, v \equiv 1 \bmod 4$;
  - $v > w_1+w_2+1, w_1+w_2 \equiv 0 \bmod 4, v \equiv 3 \bmod 4$;
  - $v < \min(w_1-w_2-1, w_2), w_1+w_2 \equiv 0 \bmod 4, v \equiv 3 \bmod 4$;
  - $\max(w_1-w_2+1, w_2) < v < w_1, w_1+w_2 \equiv 2 \bmod 4, v \equiv 3 \bmod 4$.
- $\wedge^*\pi \oplus (\pi\otimes\tau_1)\oplus(\pi\otimes\tau_2)\oplus(\tau_1\otimes\tau_2)\oplus[1]$, *where* $\pi \in \Pi_{\mathrm{alg}}^{\mathrm{Sp}_4}(\mathrm{PGL}_4)$ *has Hodge weights* $w_1 > w_2$ *and* $\tau_1, \tau_2 \in \Pi_{\mathrm{alg}}^{\perp}(\mathrm{PGL}_2)$ *have motivic weights* $v_1 > v_2$ *respectively satisfying one of the following conditions:*
  - $v_2 < w_2 < v_1$ *and* $w_1-w_2-v_2 < v_1 < w_1-w_2+v_2$;
  - $w_2 < v_2 < w_1$ *and* $v_1 > w_1+w_2+v_2$;
  - $v_2 < w_1 < v_1 < w_1-w_2+v_2$.

Proof. We take a set of generators $\{\sigma = (1,1,-1), \sigma_1 = (-1,1,1)\}$ of $\mathrm{C}_\psi = \mathrm{Z}(H) \simeq \mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$. Let χ_1,χ_2 be two generators of the character group of C_ψ such that $\chi_1(\sigma)=\chi_2(\sigma_1) = -1$ and $\chi_1(\sigma_1)=\chi_2(\sigma)=1$.

Case (i): $\psi = \wedge^*\pi \oplus (\pi\otimes\tau)\oplus\pi[2]\oplus\tau[2]\oplus[1]$, where $\pi \in \Pi_{\mathrm{alg}}^{\mathrm{Sp}_4}(\mathrm{PGL}_4)$ has Hodge weights $w_1 > w_2 > 1$ and $\tau \in \Pi_{\mathrm{alg}}^{\perp}(\mathrm{PGL}_2)$ has motivic weight v. Here we assume that Arthur's $\mathrm{SL}_2(\mathbb{C})$ is sent to the first A_1-factor of $H_{\mathbb{C}}$. In this case, the restriction of $\mathfrak{f}_4$ along ψ is isomorphic to

$$\mathrm{Sym}^2\pi \oplus (\wedge^*\pi\otimes\tau[2])\oplus(\pi\otimes\tau)\oplus\pi[2]\oplus\mathrm{Sym}^2\tau\oplus[3].$$

By Proposition 5.6.4 we have:

$$\varepsilon_\psi(\sigma) = \varepsilon(\pi) = \varepsilon(\mathbf{I}_{w_1}+\mathbf{I}_{w_2}) = (-1)^{(w_1+w_2)/2+1},$$
$$\varepsilon_\psi(\sigma_1) = \varepsilon(\wedge^*\pi\otimes\tau) = (-1)^{\max(w_1+w_2,v)+\max(w_1-w_2,v)+(v+1)/2}.$$

On the other side, the group C_ψ acts on $\wedge^*\pi, \pi\otimes\tau, \pi[2], \tau[2]$ by $1, \chi_1\chi_2, \chi_1, \chi_2$ respectively. The largest element μ_1 must be $\frac{w_1+w_2}{2}$ or $\frac{w_1+v}{2}$.

(1) If $w_2 > v$, then $\mu_1 = \frac{w_1+w_2}{2}$. Now $\mu_4 = \frac{w_1\pm1}{2}$ and $\rho_\psi^\vee = \chi_1$.
(2) If $w_2 < v$, then $\mu_1 = \frac{w_1+v}{2}$. Now μ_2 is $\frac{w_1+w_2}{2}$ or $\frac{w_2+v}{2}$.
   (a) If $w_1 > v$, then $\mu_2 = \frac{w_1+w_2}{2}$. Now $\mu_4 = \frac{w_1\pm1}{2}$ and $\rho_\psi^\vee = \chi_2$.
   (b) If $w_1 < v$, then $\mu_2 = \frac{w_2+v}{2}$. Now $\mu_4 = \frac{v\pm1}{2}$ and $\rho_\psi^\vee = \chi_1$.

By Arthur's multiplicity formula, $\mathrm{m}(\pi_\psi) = 1$ if and only if π and τ satisfy one of the conditions listed in the proposition.

Case (ii): $\psi = \wedge^*\pi \oplus (\pi\otimes\tau_1)\oplus(\pi\otimes\tau_2)\oplus(\tau_1\otimes\tau_2)\oplus[1]$, where $\pi \in \Pi_{\mathrm{alg}}^{\mathrm{Sp}_4}(\mathrm{PGL}_4)$ has Hodge weights $w_1 > w_2$ and $\tau_1,\tau_2 \in \Pi_{\mathrm{alg}}^{\perp}(\mathrm{PGL}_2)$ have motivic weights $v_1 > v_2$ respectively. In this case ε_ψ is a trivial character. On the other side, since C_ψ acts on four non-trivial irreducible summands of ψ by four different characters, $\rho_\psi^\vee = 1$ if and only if μ_1 and μ_4 come from the same irreducible summand. Now μ_1 must be $\frac{w_1+w_2}{2}$ or $\frac{w_1+v_1}{2}$ or $\frac{v_1+v_2}{2}$.

(1) If $w_2 > v_1$, then $\mu_1 = \frac{w_1+w_2}{2}$ and μ_4 can not be $\frac{w_1-w_2}{2}$, thus $\rho_\psi^\vee$ is not trivial.

(2) If $v_1 > w_2$ and $w_1 > v_2$, then $\mu_1 = \frac{w_1+v_1}{2}$. Now $\rho_\psi^\vee$ is trivial if and only if $\mu_4 = \frac{w_2+v_1}{2}$ or $\frac{v_1-w_2}{2}$.

   (a) $\mu_4 = \frac{v_1-w_2}{2}$ is equivalent to that $v_1 - w_2 > \max(v_1 - v_2, w_1 + w_2, w_1 + v_2)$. This holds if and only if $v_2 > w_2$ and $v_1 > w_1 + w_2 + v_2$.

   (b) $\mu_4 = \frac{w_2+v_1}{2}$ is equivalent to that $w_2 + v_1 > \max(w_1 - w_2, w_1 - v_2)$ and $w_2 + v_1$ is smaller than exactly two of $\{w_1 + w_2, v_1 + v_2, w_1 + v_2\}$. This holds in two cases: $w_1 < v_1 < w_1 - w_2 + v_2$ or

$$w_2 > v_2, w_1 > v_1, w_1 - w_2 - v_2 < v_1 < w_1 - w_2 + v_2.$$

(3) If $v_2 > w_1$, $\mu_1 = \frac{v_1+v_2}{2}$. We have

$$\frac{v_1 - v_2}{2} < \frac{v_1 - w_1}{2} < \frac{v_1 - w_2}{2} < \frac{v_1 + w_2}{2} < \frac{v_1 + w_1}{2} < \frac{v_1 + v_2}{2},$$

thus μ_4 can not be $\frac{v_1-v_2}{2}$ and $\rho_\psi^\vee$ is not trivial.

In conclusion, by Arthur's multiplicity formula $\mathrm{m}(\pi_\psi) = 1$ if and only if one of the following conditions holds:

- $v_2 < w_2 < v_1$ and $w_1 - w_2 - v_2 < v_1 < w_1 - w_2 + v_2$;
- $w_2 < v_2 < w_1$ and $v_1 > w_1 + w_2 + v_2$;
- $v_2 < w_1 < v_1 < w_1 - w_2 + v_2$.

□

6.3.11 $H = \left(\mathrm{A}_1^{[2^6,1^{14}]} \times \mathrm{Sp}(3)\right)/\mu_2^\Delta$

By §4.6.3, the restriction of the 26-dimensional irreducible representation J_0 of F_4 to H is isomorphic to

$$\mathrm{St} \otimes \mathrm{V}_6 + \mathbf{1} \otimes \mathrm{V}_{14},$$

where V_6 is the standard 6-dimensional representation of $\mathrm{Sp}(3)$, $\mathrm{V}_{14} = \wedge^*\mathrm{V}_6$ is the 14-dimensional irreducible representation of $\mathrm{Sp}(3)$ that is a sub-representation of $\wedge^2\mathrm{V}_6$. The centralizer of H in F_4 is $\mathrm{Z}(H) \simeq \mathbb{Z}/2\mathbb{Z}$.

For any $\pi \in \Pi_{\mathrm{alg}}^{\mathrm{Sp}_6}(\mathrm{PGL}_6)$, we denote by $\wedge^*\pi$ the representation in $\Pi_{\mathrm{alg,reg}}^{\mathrm{o}}(\mathrm{PGL}_{14})$ corresponding to the following irreducible representation of $\mathcal{L}_\mathbb{Z}$:

$$\mathcal{L}_\mathbb{Z} \xrightarrow{\psi_\pi} \mathrm{Sp}(3) \xrightarrow{\wedge^*} \mathrm{SL}_{14}(\mathbb{C}).$$

For $\psi \in \Psi_{\mathrm{AJ}}(\mathbf{F}_4)$ satisfying $\mathrm{H}(\psi) = H$ and $\mathrm{m}(\pi_\psi) = 1$, there are two possible endoscopic types:

(i) $(2, (14, 1), (6, 2))$. A global Arthur parameter of this type is of the form:

$$\wedge^*\pi \oplus \pi[2],\ \pi \in \Pi_{\mathrm{alg}}^{\mathrm{Sp}_6}(\mathrm{PGL}_6).$$

(ii) $(2, (14, 1), (12, 1))$. A global Arthur parameter of this type is of the form:

$$\wedge^*\pi \oplus (\pi \otimes \tau),\ \pi \in \Pi_{\mathrm{alg}}^{\mathrm{Sp}_6}(\mathrm{PGL}_6), \tau \in \Pi_{\mathrm{alg}}^{\perp}(\mathrm{PGL}_2).$$

For this subgroup H of F_4, the restriction of the adjoint representation $\mathfrak{f}_4$ of F_4 to H is isomorphic to

$$\mathrm{Sym}^2\,\mathrm{St} \otimes \mathbf{1} + \mathrm{St} \otimes \mathrm{V}_{14}' + \mathbf{1} \otimes \mathrm{Sym}^2\,\mathrm{V}_6,$$

where V_{14}' is another 14-dimensional irreducible representation of $\mathrm{Sp}(3)$ that is not equivalent to $\mathrm{V}_{14} = \wedge^*\mathrm{V}_6$.

Proposition* 6.3.15. *For a discrete global Arthur parameter $\psi \in \Psi_{\mathrm{AJ}}(\mathbf{F}_4)$ satisfying $\mathrm{H}(\psi) = H$, the multiplicity $\mathrm{m}(\pi_\psi) = 1$ if and only if ψ is one of the following parameters:*

- $\wedge^*\pi \oplus \pi[2]$, *where $\pi \in \Pi_{\mathrm{alg}}^{\mathrm{Sp}_6}(\mathrm{PGL}_6)$ has Hodge weights $w_1 > w_2 > w_3 > 1$ and one of the following conditions holds:*
  - $w_1 > w_2 + w_3 + 1$ *and* $w_1 + w_2 + w_3 \equiv 3 \bmod 4$*;*
  - $w_1 < w_2 + w_3 - 1$ *and* $w_1 + w_2 + w_3 \equiv 1 \bmod 4$*.*
- $\wedge^*\pi \oplus (\pi \otimes \tau)$, *where $\pi \in \Pi_{\mathrm{alg}}^{\mathrm{Sp}_6}(\mathrm{PGL}_6)$ has Hodge weights $w_1 > w_2 > w_3$ and $\tau \in \Pi_{\mathrm{alg}}^{\perp}(\mathrm{PGL}_2)$ has motivic weight v satisfying one of the following conditions:*
  - $|w_1 - w_2 - w_3| < v < w_3$*;*
  - $w_1 - w_2 + w_3 < v < w_2$*;*
  - $w_3 < v < \min(w_2, w_1 - w_2 - w_3)$*;*
  - $\max(w_2, w_1 - w_2 - w_3) < v < w_1 - w_2 + w_3$*;*
  - $w_1 < v < w_1 + w_2 - w_3$*;*
  - $v > w_1 + w_2 + w_3$*.*

Proof. We denote the generator $(-1, 1) \in \mathrm{Z}(H) = \mathrm{C}_\psi$ by γ.

Case (i): $\psi = \wedge^*\pi \oplus \pi[2]$, where $\pi \in \Pi_{\mathrm{alg}}^{\mathrm{Sp}_6}(\mathrm{PGL}_6)$ has Hodge weights $w_1 > w_2 > w_3 > 1$. In this case, the restriction of $\mathfrak{f}_4$ along ψ is isomorphic to

$$\mathrm{Sym}^2\,\pi \oplus \pi'[2] \oplus [3],$$

where $\pi' \in \Pi_{\mathrm{alg}}^{\perp}(\mathrm{PGL}_{14})$ corresponds to

$$\mathcal{L}_{\mathbb{Z}} \xrightarrow{\psi_\pi} \mathrm{Sp}(3) \xrightarrow{\mathrm{V}_{14}'} \mathrm{SL}_{14}(\mathbb{C}).$$

Notice that $\wedge^3\mathrm{V}_6 \simeq \mathrm{V}_{14}' \oplus \mathrm{V}_6$, thus the Hodge weights of π' are

$$\pm w_1, \pm w_2, \pm w_3, \pm w_1 \pm w_2 \pm w_3.$$

By Proposition 5.6.4 we have:

$$\begin{aligned}\varepsilon_\psi(\gamma) &= \varepsilon\left(\mathbf{I}_{w_1} + \mathbf{I}_{w_2} + \mathbf{I}_{w_3} + \mathbf{I}_{w_1+w_2+w_3} + \mathbf{I}_{w_1+w_2-w_3} + \mathbf{I}_{w_1-w_2+w_3} + \mathbf{I}_{|w_1-w_2-w_3|}\right) \\ &= (-1)^{(w_1+w_2+w_3+1)/2+\max(w_1,w_2+w_3)}.\end{aligned}$$

On the other side, γ acts on $\wedge^*\pi$ by 1 and on $\pi[2]$ by -1. The largest element μ_1 must be $\frac{w_1+w_2}{2}$. Now $\mu_4 = \frac{w_1 \pm 1}{2}$, thus $\rho_\psi^\vee(\gamma) = -1$. By Arthur's multiplicity formula, $\mathrm{m}(\pi_\psi) = 1$ if and only if one of the following conditions holds:

- $w_1 > w_2 + w_3 + 1$ and $w_1 + w_2 + w_3 \equiv 3 \bmod 4$;
- $w_1 < w_2 + w_3 - 1$ and $w_1 + w_2 + w_3 \equiv 1 \bmod 4$.

Case (ii): $\psi = \wedge^* \pi \oplus (\pi \otimes \tau)$, where $\pi \in \Pi_{\mathrm{alg}}^{\mathrm{Sp}_6}(\mathrm{PGL}_6)$ has Hodge weights $w_1 > w_2 > w_3$ and $\tau \in \Pi_{\mathrm{alg}}^{\perp}(\mathrm{PGL}_2)$ has motivic weight v. In this case ε_ψ is trivial. On the other side, the largest μ_1 must be $\frac{w_1+w_2}{2}$ or $\frac{w_1+v}{2}$.

(1) If $v < w_2$, then $\mu_1 = \frac{w_1+w_2}{2}$.

  (a) If $v < w_3$, then μ_4 is the middle one in $\{\frac{w_2+w_3}{2}, \frac{w_1+v}{2}, \frac{w_1-v}{2}\}$. Hence $\rho_\psi^\vee = 1$ if and only if $\mu_4 = \frac{w_2+w_3}{2}$, which is equivalent to $v > |w_1 - w_2 - w_3|$.

  (b) If $v > w_3$, then μ_4 is the middle one in $\{\frac{w_2+v}{2}, \frac{w_1+w_3}{2}, \frac{w_1-w_3}{2}\}$. Hence $\rho_\psi^\vee = 1$ if and only if $\mu_4 = \frac{w_1 \pm w_3}{2}$, which is equivalent to $v > w_1 - w_2 + w_3$ or $v < w_1 - w_2 - w_3$.

(2) If $v > w_2$, then $\mu_1 = \frac{w_1+v}{2}$.

  (a) If $v < w_1$, then μ_4 is the middle one in $\{\frac{w_2+v}{2}, \frac{w_1+w_3}{2}, \frac{w_1-w_3}{2}\}$. Hence $\rho_\psi^\vee = 1$ if and only if $\mu_4 = \frac{w_2+v}{2}$, which is equivalent to $w_1 - w_2 - w_3 < v < w_1 - w_2 + w_3$.

  (b) If $v > w_1$, then μ_4 is the middle one in $\{\frac{w_1+w_2}{2}, \frac{v+w_3}{2}, \frac{v-w_3}{2}\}$. Hence $\rho_\psi^\vee = 1$ if and only if $\mu_4 = \frac{v \pm w_3}{2}$, which is equivalent to $v > w_1 + w_2 + w_3$ or $v < w_1 + w_2 - w_3$.

In conclusion, $\mathrm{m}(\pi_\psi) = 1$ if and only if one of the conditions on π, τ listed in the proposition is satisfied. □

6.3.12 $H = \mathrm{Spin}(8)$

By §4.6.5, the restriction of the 26-dimensional irreducible representation J_0 of F_4 to H is isomorphic to

$$\mathbf{1}^{\oplus 2} + \mathrm{V}_8 + \mathrm{V}_{\mathrm{Spin}}^{+} + \mathrm{V}_{\mathrm{Spin}}^{-},$$

where V_8 is the 8-dimensional vector representation of $\mathrm{Spin}(8)$, i.e. the composition of $\mathrm{Spin}(8) \to \mathrm{SO}(8)$ with the standard 8-dimensional representation of $\mathrm{SO}(8)$, and $\mathrm{V}_{\mathrm{Spin}}^{\pm}$ are two 8-dimensional spinor representations. The centralizer of H in F_4 is $\mathrm{Z}(H) \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

For $\psi \in \Psi_{\mathrm{AJ}}(\mathbf{F}_4)$ satisfying $\mathrm{H}(\psi) = H$ and $\mathrm{m}(\pi_\psi) = 1$, there is only one possible endoscopic type: $(5, (8,1), (8,1), (8,1), (1,1), (1,1))$. A global Arthur parameter of this type is of the form:

$$\psi = \pi \oplus \mathrm{Spin}^+ \pi \oplus \mathrm{Spin}^- \pi \oplus [1] \oplus [1],\ \pi \in \Pi_{\mathrm{alg}}^{\mathrm{SO}_8}(\mathrm{PGL}_8),$$

where we lift $\psi_\pi : \mathcal{L}_\mathbb{Z} \twoheadrightarrow \mathrm{SO}(8) \to \mathrm{SO}_8(\mathbb{C})$ to $\widetilde{\psi_\pi} : \mathcal{L}_\mathbb{Z} \to \mathrm{Spin}_8(\mathbb{C})$, and $\mathrm{Spin}^* \pi, * = \pm$ is the representation corresponding to

$$\mathcal{L}_\mathbb{Z} \xrightarrow{\widetilde{\psi_\pi}} \mathrm{Spin}_8(\mathbb{C}) \xrightarrow{V_{\mathrm{Spin}}^*} \mathrm{SL}_8(\mathbb{C}).$$

Proposition* 6.3.16. *For any discrete global Arthur parameter $\psi \in \Psi_{\mathrm{AJ}}(\mathbf{F}_4)$ satisfying $\mathrm{H}(\psi) = H$, we have $\mathrm{m}(\pi_\psi) = 0$.*

Proof. Let $\psi = \pi \oplus \mathrm{Spin}^+ \pi \oplus \mathrm{Spin}^- \pi \oplus [1] \oplus [1]$, where $\pi \in \Pi_{\mathrm{alg}}^{\mathrm{SO}_8}(\mathrm{PGL}_8)$ has Hodge weights $2w_1 > 2w_2 > 2w_3 > 2w_4$. The global component group $\mathrm{C}_\psi \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ and it acts on $\pi, \mathrm{Spin}^+ \pi, \mathrm{Spin}^- \pi$ by three different characters.

Since ε_ψ is trivial, by Arthur's trace formula $\mathrm{m}(\pi_\psi) = 1$ if and only if $\rho_\psi^\vee = 1$, which is equivalent to that μ_1 and μ_4 come from the same irreducible summand of ψ by Proposition 6.2.1.

In this case, the largest element μ_1 must be w_1 or $\frac{w_1+w_2+w_3+w_4}{2}$.

(1) If $w_1 > w_2 + w_3 + w_4$, then $\mu_1 = w_1$. Now we have

$$w_2 < \frac{w_1 + w_2 - w_3 + w_4}{2} < \frac{w_1 + w_3 + w_3 - w_4}{2} < \frac{w_1 + w_2 + w_3 + w_4}{2} < \mu_1,$$

thus μ_4 does not come from π. Hence $\rho_\psi^\vee$ is not trivial.

(2) If $w_1 < w_2 + w_3 + w_4$, then $\mu_1 = \frac{w_1+w_2+w_3+w_4}{2}$. Now we have

$$\frac{w_1 - w_2 + w_3 - w_4}{2} < \frac{w_1 + w_2 - w_3 - w_4}{2} < \min\left(w_2, \frac{w_1 + w_2 \pm (w_3 - w_4)}{2}\right) < \mu_1$$

and

$$\frac{|w_1 - w_2 - w_3 + w_4|}{2} \leq \max\left(w_4, \frac{-w_1 + w_2 + w_3 + w_4}{2}\right)$$

is also smaller than at least 4 weights, hence

$$\mu_4 \notin \left\{\frac{w_1 - w_2 + w_3 - w_4}{2}, \frac{w_1 + w_2 - w_3 - w_4}{2}, \frac{|w_1 - w_2 - w_3 + w_4|}{2}\right\}.$$

So μ_4 does not come from $\mathrm{Spin}^+ \pi$ and $\rho_\psi^\vee$ is not trivial.

In conclusion, by Arthur's multiplicity formula the multiplicity $\mathrm{m}(\pi_\psi)$ is always 0. □

6.3.13 $H = \mathrm{Spin}(9)$

By §4.6.2, the restriction of the 26-dimensional irreducible representation J_0 of F_4 to H is isomorphic to

$$\mathbf{1} + \mathrm{V}_9 + \mathrm{V}_{\mathrm{Spin}},$$

where V_9 is the standard representation of $\mathrm{Spin}(9)$, $\mathrm{V}_{\mathrm{Spin}}$ is the 16-dimensional spinor representations. The centralizer of H in F_4 is $\mathrm{Z}(H) \simeq \mathbb{Z}/2\mathbb{Z}$.

For $\psi \in \Psi_{\mathrm{AJ}}(\mathbf{F}_4)$ satisfying $\mathrm{H}(\psi) = H$ and $\mathrm{m}(\pi_\psi) = 1$, there is only one possible endoscopic type: $(3, (16, 1), (9, 1), (1, 1))$. A global Arthur parameter of this type is of the form:

$$\psi = \pi \oplus \mathrm{Spin}\,\pi \oplus [1],\ \pi \in \Pi_{\mathrm{alg}}^{\mathrm{SO}_9}(\mathrm{PGL}_9),$$

where we lift $\psi_\pi : \mathcal{L}_\mathbb{Z} \twoheadrightarrow \mathrm{SO}(9) \to \mathrm{SO}_9(\mathbb{C})$ to $\widetilde{\psi_\pi} : \mathcal{L}_\mathbb{Z} \to \mathrm{Spin}_9(\mathbb{C})$, and $\mathrm{Spin}\,\pi$ is the representation corresponding to

$$\mathcal{L}_\mathbb{Z} \xrightarrow{\widetilde{\psi_\pi}} \mathrm{Spin}_9(\mathbb{C}) \xrightarrow{V_{\mathrm{Spin}}} \mathrm{SL}_{16}(\mathbb{C}).$$

Proposition* 6.3.17. *A discrete global Arthur parameter $\psi \in \Psi_{\mathrm{AJ}}(\mathbf{F}_4)$ satisfies $\mathrm{H}(\psi) = H$ and $\mathrm{m}(\pi_\psi) = 1$ if and only if $\psi = \pi \oplus \mathrm{Spin}\,\pi \oplus [1]$, where $\pi \in \Pi_{\mathrm{alg}}^{\mathrm{SO}_9}(\mathrm{PGL}_9)$ has Hodge weights $w_1 > w_2 > w_3 > w_4$ satisfying $w_2 + w_3 - w_4 < w_1 < w_2 + w_3 + w_4$.*

Proof. Let $\psi = \pi \oplus \mathrm{Spin}\,\pi \oplus [1]$, where $\pi \in \Pi_{\mathrm{alg}}^{\mathrm{SO}_9}(\mathrm{PGL}_9)$ has Hodge weights $w_1 > w_2 > w_3 > w_4$. The global component group C_ψ is a cyclic 2-group, and it acts on π trivially and on $\mathrm{Spin}\,\pi$ by its non-trivial character.

Since the parameter is tempered, ε_ψ is trivial. By Arthur's multiplicity formula, $\mathrm{m}(\pi_\psi) = 1$ if and only if $\rho_\psi^\vee = 1$, which is equivalent to that μ_1 and μ_4 come from the same irreducible summand of ψ by Proposition 6.2.1. In this case, the largest element $\mu_1 = \frac{w_1}{2}$ or $\frac{w_1+w_2+w_3+w_4}{4}$.

(1) If $w_1 > w_2+w_3+w_4$, then $\mu_1 = \frac{w_1}{2}$. By our discussion in the proof of Proposition 6.3.16, μ_4 does not come from π, thus $\rho_\psi^\vee$ is not trivial.

(2) If $w_1 < w_2 + w_3 + w_4$, then $\mu_1 = \frac{w_1+w_2+w_3+w_4}{4}$. Now $\mu_4 = \max\left(\frac{w_2}{2}, \frac{w_1+w_2-w_3+w_4}{4}\right)$. Hence $\rho_\psi^\vee$ is trivial if and only if $w_1 + w_4 > w_2 + w_3$.

In conclusion, $\mathrm{m}(\pi_\psi) = 1$ if and only if $w_2 + w_3 - w_4 < w_1 < w_2 + w_3 + w_4$. □

6.3.14 $H = \mathrm{F}_4$

For stable tempered parameters, the component group is trivial and as a direct consequence we have:

Proposition* 6.3.18. *For any discrete global Arthur parameter $\psi \in \Psi_{\mathrm{AJ}}(\mathbf{F}_4)$ satisfying $\mathrm{H}(\psi) = \mathrm{F}_4$, we have $\mathrm{m}(\pi_\psi) = 1$.*

6.4 Classification of representations contributing to $\mathcal{A}_{\mathrm{V}_\lambda}(\mathbf{F}_4)$

Recall that in §5.1, for each irreducible representation V_λ with highest weight λ of $\mathrm{F}_4 = \mathbf{F}_4(\mathbb{R})$, we have defined its multiplicity space in $\mathcal{L}_{\mathrm{disc}}(\mathbf{F}_4)$:

$$\mathcal{A}_{\mathrm{V}_\lambda}(\mathbf{F}_4) = \mathrm{Hom}_{\mathbf{F}_4(\mathbb{R})}(\mathrm{V}_\lambda, \mathcal{L}_{\mathrm{disc}}(\mathbf{F}_4)^{\mathcal{F}_{4,\mathrm{I}}(\widehat{\mathbb{Z}})}),$$

which parametrizes level one discrete automorphic representation π of $\mathbf{F}_4$ such that $\pi_\infty \simeq \mathrm{V}_\lambda$. We have a dimension formula Corollary 5.1.8 for this space. Now with results in §6.3, we can study the discrete global Arthur parameters $\psi \in \Psi_{\mathrm{AJ}}(\mathbf{F}_4)$ whose corresponding representation $\pi_\psi \in \Pi(\mathbf{F}_4)$ has multiplicity 1 in $\mathcal{L}_{\mathrm{disc}}(\mathbf{F}_4)$ and contributes to $\mathcal{A}_{\mathrm{V}_\lambda}(\mathbf{F}_4)$.

According to Lemma 5.1.5, we have:

$$\dim \mathcal{A}_{\mathrm{V}_\lambda}(\mathbf{F}_4) = \sum_{\pi \in \Pi(\mathbf{F}_4),\, \pi_\infty \simeq \mathrm{V}_\lambda} \mathrm{m}(\pi).$$

Using discrete global Arthur parameters, we rewrite this formula as

$$\dim \mathcal{A}_{\mathrm{V}_\lambda}(\mathbf{F}_4) = \sum_{\psi \in \Psi_{\mathrm{AJ}}(\mathbf{F}_4),\, \mathrm{c}_\infty(\psi) = \mathrm{c}_\infty(\mathrm{V}_\lambda)} \mathrm{m}(\pi_\psi) = \sum_{\psi \in \Psi_{\mathrm{AJ}}(\mathbf{F}_4),\, \mathrm{c}_\infty(\psi) = \lambda + \rho} \mathrm{m}(\pi_\psi),$$

where ρ is the half sum of positive roots of $\mathbf{F}_4$.

If the endoscopic type of $\psi \in \Psi_{\mathrm{AJ}}(\mathbf{F}_4)$ is not stable, i.e. $\mathrm{H}(\psi)$ is the conjugacy class of a proper subgroup of $\mathrm{F}_4 = \mathbf{F}_4(\mathbb{R})$, then it must have one of the types listed in §6.3. For each subgroup H of F_4 listed in Theorem 4.6.7, we can determine the discrete global Arthur parameters $\psi \in \Psi_{\mathrm{AJ}}(\mathbf{F}_4)$ satisfying $\mathrm{H}(\psi) = H$ and $\mathrm{m}(\pi_\psi) = 1$. The difference

$$\dim \mathcal{A}_{\mathrm{V}_\lambda}(\mathbf{F}_4) - \#\{\psi \in \Psi_{\mathrm{AJ}}(\mathbf{F}_4) \,|\, \mathrm{H}(\psi) \neq \mathrm{F}_4, \mathrm{c}_\infty(\psi) = \rho + \lambda, \mathrm{m}(\pi_\psi) = 1\} \qquad (6.2)$$

is the number of discrete automorphic representations π of $\mathbf{F}_4$ with archimedean component $\pi_\infty \simeq \mathrm{V}_\lambda$ whose global Arthur parameter is tempered and stable. In other words:

Proposition* 6.4.1. *Let λ be a dominant weight of F_4, we define the number*

$$\mathrm{F}_4(\lambda) := \#\{\pi \in \Pi_{\mathrm{cusp}}(\mathrm{PGL}_{26}) \,|\, \mathrm{c}_\infty(\pi) = \mathrm{r}_0(\lambda + \rho) \in \mathfrak{sl}_{26,\mathrm{ss}}, \mathrm{H}(\pi) \simeq \mathrm{F}_4\},$$

where $\mathrm{r}_0 : \mathfrak{f}_4 \to \mathfrak{sl}_{26}$ is the 26-dimensional irreducible representation of $\mathfrak{f}_4$, and define $\mathrm{w}(\lambda)$ to be twice the maximal eigenvalue of $\lambda + \rho$. Then we have a formula for the number $\mathrm{F}_4(\lambda)$, and we list nonzero $\mathrm{F}_4(\lambda)$ for all the dominant weights λ such that $\mathrm{w}(\lambda) \leq 44$ in Table 11.

Proof. The formula for $\mathrm{F}_4(\lambda)$ follows from (6.2) and our classifications in §6.3. This formula involves the numbers of elements in one of the following sets with certain Hodge weights:

$$\Pi_{\mathrm{alg}}^{\perp}(\mathrm{PGL}_2), \Pi_{\mathrm{alg}}^{\mathrm{Sp}_4}(\mathrm{PGL}_4), \Pi_{\mathrm{alg}}^{\mathrm{Sp}_6}(\mathrm{PGL}_6), \Pi_{\mathrm{alg}}^{\mathrm{G}_2}(\mathrm{PGL}_7), \Pi_{\mathrm{alg}}^{\mathrm{SO}_9}(\mathrm{PGL}_9).$$

For $\Pi_{\mathrm{alg}}^{\perp}(\mathrm{PGL}_2)$, this number is related to the dimension of cusp forms for $\mathrm{SL}_2(\mathbb{Z})$, as explained in Example 5.4.6. For other four sets, we can find some tables in [CR] and [CT]. A [PARI/GP] program to compute $\mathrm{F}_4(\lambda)$ for dominant weights λ satisfying $\mathrm{w}(\lambda) \leq 60$ is provided in [Sha]. □

Remark 6.4.2. The formula for $\mathrm{F}_4(\lambda)$ has too many terms, thus it is not reasonable to write it down here. However, under some hypothesis on λ, many terms vanish and this formula becomes much simpler. For example, if

- $\lambda_i > 0$ for $i = 1, 2, 3, 4$,
- $\lambda_1 > \lambda_2 + \lambda_3 + \lambda_4 + 3$,
- and λ_3, λ_4 are both odd,

then we have the following formula:

$$\mathrm{F}_4(\lambda) = \dim \mathcal{A}_{\mathrm{V}_\lambda}(\mathbf{F}_4) - \mathrm{O}^*(\lambda_1', \lambda_2', \lambda_3', \lambda_4'),$$

where $\mathrm{O}^*(w_1, w_2, w_3, w_4)$ is the number of equivalence classes of level one cuspidal orthogonal representations of PGL_9 with Hodge weights $w_1 > w_2 > w_3 > w_4 > 0$, and

$$\lambda_1' = 2\lambda_1 + 6\lambda_2 + 4\lambda_3 + 2\lambda_4 + 14,\ \lambda_2' = 2\lambda_1 + 2\lambda_2 + 2\lambda_3 + 2\lambda_4 + 8,$$
$$\lambda_3' = 2\lambda_1 + 2\lambda_2 + 2\lambda_3 + 6,\ \lambda_4' = 2\lambda_1 + 2\lambda_2 + 4.$$

In Table 11, we find that the smallest $\mathrm{w}(\lambda)$ for λ such that $\mathrm{F}_4(\lambda) \neq 0$ is 36 and the corresponding dominant weight is $\lambda = \varpi_1 + 2\varpi_2 + 2\varpi_4$. We are now going to prove this fact without using Theorem 6.3.1, in order to give readers who skip the proof of Theorem 6.3.1 an example of how we apply Arthur's conjectures.

Proposition* 6.4.3. *There is a level one cuspidal automorphic representation* π *of* PGL_{26} *with motivic weight* 36*, such that the Sato-Tate group* $\mathrm{H}(\pi)$ *of* π *is isomorphic to the compact Lie group* F_4*.*

Proof. We fix $\lambda = \varpi_1 + 2\varpi_2 + 2\varpi_4$. In Table 6, we find that $\dim \mathcal{A}_{\mathrm{V}_\lambda}(\mathbf{F}_4) = 1$. We denote the unique automorphic representation contributing to $\mathcal{A}_{\mathrm{V}_\lambda}(\mathbf{F}_4)$ by π_0 and its corresponding discrete global Arthur parameter by ψ_0. The eigenvalues of $\mathrm{c}_\infty(\pi_0) = \lambda + \rho$ are:

$$-18, -16, -13, -12, -9, -9, -7, -6, -5, -4, -3, -2, 0, 0, 2, 3, 4, 5, 6, 7, 9, 9, 12, 13, 16, 18.$$

Now it suffices to show that $\mathrm{H}(\psi_0) = \mathrm{F}_4$.

We can exclude some possibilities of $\mathrm{H}(\psi_0)$ and endoscopic types by an argument of motivic weights. For example, if $\mathrm{H}(\psi_0) = \mathrm{A}_1^{[17,9]}$ and $\psi_0 = \mathrm{Sym}^{16}\pi \oplus \mathrm{Sym}^8\pi$ for some $\pi \in \Pi_{\mathrm{alg}}^{\perp}(\mathrm{PGL}_2)$, then $\mathrm{w}(\pi_0) = 16\mathrm{w}(\pi) \geq 16 \times 11 = 176$, which contradicts with $\mathrm{w}(\pi_0) = 36$. We also notice that 1 is not an eigenvalue of $\mathrm{c}_\infty(\pi_0)$, thus ψ_0 does not have irreducible summands of the form

$$\pi[d], \text{ where } \pi \in \Pi_{\mathrm{alg}}^{\perp}(\mathrm{PGL}_n), n \equiv 1 \bmod 2 \text{ and } d \geq 3.$$

Now we list all possible types for ψ_0:

(1) ψ_0 is a stable and tempered parameter;
(2) $\psi_0 = (\bigoplus_{1\leq i<j\leq 3} \pi_i \otimes \pi_j) \oplus (\bigoplus_{1\leq i\leq 3} \pi_i[2]) \oplus [1] \oplus [1], \pi_1, \pi_2, \pi_3 \in \Pi_{\mathrm{alg}}^{\perp}(\mathrm{PGL}_2)$;
(3) $\psi_0 = (\bigoplus_{1\leq i<j\leq 4} \pi_i \otimes \pi_j) \oplus [1] \oplus [1], \pi_1, \pi_2, \pi_3, \pi_4 \in \Pi_{\mathrm{alg}}^{\perp}(\mathrm{PGL}_2)$;
(4) $\psi_0 = \wedge^*\pi \oplus (\pi \otimes \tau) \oplus \pi[2] \oplus \tau[2] \oplus [1], \pi \in \Pi_{\mathrm{alg}}^{\mathrm{Sp}_4}(\mathrm{PGL}_4), \tau \in \Pi_{\mathrm{alg}}^{\perp}(\mathrm{PGL}_2)$;
(5) $\psi_0 = \wedge^*\pi \oplus (\pi \otimes \tau_1) \oplus (\pi \otimes \tau_2) \oplus (\tau_1 \otimes \tau_2) \oplus [1], \pi \in \Pi_{\mathrm{alg}}^{\mathrm{Sp}_4}(\mathrm{PGL}_4), \tau_1, \tau_2 \in \Pi_{\mathrm{alg}}^{\perp}(\mathrm{PGL}_2)$;
(6) $\psi_0 = \wedge^*\pi \oplus \pi[2], \pi \in \Pi_{\mathrm{alg}}^{\mathrm{Sp}_6}(\mathrm{PGL}_6)$;
(7) $\psi_0 = \wedge^*\pi \oplus (\pi \otimes \tau), \pi \in \Pi_{\mathrm{alg}}^{\mathrm{Sp}_6}(\mathrm{PGL}_6), \tau \in \Pi_{\mathrm{alg}}^{\perp}(\mathrm{PGL}_2)$;
(8) $\psi_0 = \pi \oplus \mathrm{Spin}^+\pi \oplus \mathrm{Spin}^-\pi \oplus [1] \oplus [1], \pi \in \Pi_{\mathrm{alg}}^{\mathrm{SO}_8}(\mathrm{PGL}_8)$;
(9) $\psi_0 = \pi \oplus \mathrm{Spin}\,\pi \oplus [1], \pi \in \Pi_{\mathrm{alg}}^{\mathrm{SO}_9}(\mathrm{PGL}_9)$.

The definitions of some notations like $\wedge^*, \mathrm{Spin}^\pm$ can be found in §6.3. Now we are going to show that ψ_0 can not be of the types listed above except (1).

Type (2): The Hodge weights of the irreducible summand $\pi_i[2], i = 1, 2, 3$ are $\mathrm{w}(\pi_i) \pm 1$, thus there are two consecutive integers $\frac{\mathrm{w}(\pi_i)\pm 1}{2}$ in the eigenvalues of $\mathrm{c}_\infty(\pi_0)$. The possible $\mathrm{w}(\pi_i)$'s are $5, 7, 9, 11, 13, 25$. However, $\Pi_{\mathrm{alg}}^{\perp}(\mathrm{PGL}_2)$ contains no representations with motivic weights $5, 7, 9, 13$, thus we are unable to find three different $\mathrm{w}(\pi_i)$. If $\pi_i \simeq \pi_j$ for some i, j, then $\pi_i \otimes \pi_j$ has two zero weights, which is a contradiction!

Type (3): By the same argument for type (2), ψ_0 can not be of this type.

Type (4): Denote the Hodge weights of $\pi \in \Pi_{\mathrm{alg}}^{\mathrm{Sp}_4}(\mathrm{PGL}_4)$ by $w_1 > w_2$. By a similar argument for type (2), we can see that $w_1, w_2 \in \{5, 7, 9, 11, 13, 25\}$. Via the help of [CR15, Table 5], we have $w_1 = 25$ and $w_2 \in \{5, 7, 9\}$, thus $\mathrm{w}(\tau)$ must be 11. Since $(w_1 + w_2)/2$ has to be an eigenvalue of $\mathrm{c}_\infty(\pi_\infty)$, the smaller Hodge weight w_2 can only be 7.

Now we use Arthur's multiplicity formula. In this case

$$\mathrm{H}(\psi_0) = \left(\mathrm{A}_1^{[2^6,1^{14}]} \times \mathrm{A}_1^{[2^6,1^{14}]} \times \mathrm{Sp}(2)\right)/\mu_2^{\Delta},$$

and by §4.6.6 the global component group $\mathrm{C}_{\psi_0} \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. We take a set of generators $\{\sigma = (1, 1, -1), \sigma_1 = (-1, 1, 1)\}$ of C_{ψ_0}. The restriction of the adjoint representation $\mathfrak{f}_4$ of F_4 along ψ_0 is isomorphic to

$$\mathrm{Sym}^2 \pi \oplus (\wedge^* \pi \otimes \tau[2]) \oplus (\pi \otimes \tau) \oplus \pi[2] \oplus \mathrm{Sym}^2 \tau \oplus [3].$$

By Proposition 5.6.4 we have:

$$\varepsilon_{\psi_0}(\sigma) = \varepsilon(\pi) = \varepsilon(\mathbf{I}_7) \cdot \varepsilon(\mathbf{I}_{25}) = -1.$$

On the other side $\mu_1 = 36$ comes from $\pi \otimes \tau$ and $\mu_4 = 24$ comes from $\pi[2]$. The element σ acts on $\pi \otimes \tau$ and $\pi[2]$ both by -1, thus $\rho^\vee_{\psi_0}(\sigma) = 1$ by Proposition 6.2.1. By Arthur's multiplicity formula, the corresponding representation has multiplicity 0 in $\mathcal{L}_{\mathrm{disc}}(\mathbf{F}_4)$.

Type (5): Denote the Hodge weights of $\pi \in \Pi^{\mathrm{Sp}_4}_{\mathrm{alg}}(\mathrm{PGL}_4)$ by $w_1 > w_2$, and assume that $\mathrm{w}(\tau_1) > \mathrm{w}(\tau_2)$. Since $36 \geq w_1 + \mathrm{w}(\tau_1) \geq w_1 + 15$, we have $w_1 \leq 21$, thus $(w_1, w_2) = (19, 7)$ or $(21, 5), (21, 9), (21, 13)$ by [CR15, Table 5]. We also need $(w_1 \pm w_2)/2$ to be eigenvalues of $\mathrm{c}_\infty(\pi_0)$, so $(w_1, w_2) = (19, 7)$. However, the equalities $36 = w_1 + \mathrm{w}(\tau_1)$ and $32 = w_1 + \mathrm{w}(\tau_2)$ imply that $\mathrm{w}(\tau_1) = 17, \mathrm{w}(\tau_2) = 13$, which contradicts with the non-existence of representations in $\Pi^\perp_{\mathrm{alg}}(\mathrm{PGL}_2)$ with Hodge weight 13.

Type (6): Denote the Hodge weights of $\pi \in \Pi^{\mathrm{Sp}_6}_{\mathrm{alg}}(\mathrm{PGL}_6)$ by $w_1 > w_2 > w_3$. We have three pairs of consecutive integers $\frac{w_i \pm 1}{2}$ in the eigenvalues of $\mathrm{c}_\infty(\pi_0)$, thus for $i = 1, 2, 3$ we have $w_i \in \{5, 7, 9, 11, 13, 25\}$. By [CR15, Table 6], (w_1, w_2, w_3) must be $(25, 13, 7)$. However, $\wedge^* \pi$ has 38 as its weight, which is a contradiction.

Type (7): Denote the Hodge weights of $\pi \in \Pi^{\mathrm{Sp}_6}_{\mathrm{alg}}(\mathrm{PGL}_6)$ by $w_1 > w_2 > w_3$. Since $36 \geq w_1 + \mathrm{w}(\tau) \geq w_1 + 11$, we have $23 \leq w_1 \leq 25$. Combining $36 \geq w_1 + w_2$ with [CR15, Table 6], we get $(w_1, w_2, w_3) = (23, 13, 5)$. However, $\mathrm{w}(\tau) = 32 - w_1 = 9 < 11$, which is a contradiction.

Type (8): Denote the Hodge weights of $\pi \in \Pi^{\mathrm{SO}_8}_{\mathrm{alg}}(\mathrm{PGL}_8)$ by $w_1 > w_2 > w_3 > w_4$. The multiset

$$\{\pm w_1/2, \pm w_2/2, \pm w_3/2, \pm w_4/2, \frac{\pm w_1 \pm w_2 \pm w_3 \pm w_4}{4}, 0, 0\}$$

coincides with the multiset of eigenvalues of $\mathrm{c}_\infty(\pi_0)$. The solutions to this system of equations are

$$(w_1, w_2, w_3, w_4) = (26, 24, 18, 4), (32, 18, 12, 10), (36, 14, 8, 6).$$

By the method of Chenevier-Taïbi in [CT20], there are no representations in $\Pi^{\mathrm{SO}_8}_{\mathrm{alg}}(\mathrm{PGL}_8)$ with these Hodge weights.

Type (9): By the same argument for type (9), we get the Hodge weights of $\pi \in \Pi^{\mathrm{SO}_9}_{\mathrm{alg}}(\mathrm{PGL}_9)$:

$$(w_1, w_2, w_3, w_4) = (26, 24, 18, 4), (32, 18, 12, 10), (36, 14, 8, 6).$$

Again by the method in [CT20], there are no representations in $\Pi^{\mathrm{SO}_9}_{\mathrm{alg}}(\mathrm{PGL}_9)$ with these Hodge weights.

In conclusion, the discrete global Arthur parameter ψ_0 is a stable and tempered parameter, i.e. $\mathrm{H}(\psi_0) = \mathrm{F}_4$. Composing this ψ_0 with the 26-dimensional irreducible representation $r_0 : \widehat{\mathbf{F}_4}(\mathbb{C}) \to \mathrm{SL}_{26}(\mathbb{C})$, we get an irreducible 26-dimensional representation of $\mathcal{L}_{\mathbb{Z}}$, and its corresponding cuspidal representation of PGL_{26} is the desired one. □

For each dominant weight λ of $\mathbf{F}_4$, we define $\Psi_\lambda(\mathbf{F}_4)$ to be the set

$$\{\psi \in \Psi_{\mathrm{AJ}}(\mathbf{F}_4) \,|\, \pi_\psi \in \Pi_{\mathrm{disc}}(\mathbf{F}_4) \text{ and } (\pi_\psi)_\infty \simeq \mathrm{V}_\lambda\}.$$

In Table 9 and Table 10, we list the elements of $\Psi_\lambda(\mathbf{F}_4)$ for weights λ such that $\mathrm{w}(\lambda) \leq 36$ and $\Psi_\lambda(\mathbf{F}_4) \neq \emptyset$, where we use the following notations:

Notation 6.4.4. For a representation π in $\Pi_{\mathrm{alg}}^{\mathrm{Sp}_{2n}}(\mathrm{PGL}_{2n}), n = 1, 2, 3$ with Hodge weights $w_1 > w_2 > \cdots > w_n$, we denote it by $\Delta_{w_1,\ldots,w_n}$. If there are $k \geq 1$ equivalence classes of cuspidal representations with these Hodge weights, we give them a superscript $\Delta_{w_1,\ldots,w_n}^{(k)}$, meaning that in this case we have k different choices of cuspidal representations. Similarly, for k different representations π in $\Pi_{\mathrm{alg}}^{\mathrm{SO}_9}(\mathrm{PGL}_9)$ or $\Pi_{\mathrm{alg}}^{\mathrm{G}_2}(\mathrm{PGL}_7)$ with Hodge weights $w_1 > \cdots > w_n$, where $n = 3$ or 4, we denote them by $\Delta_{w_1,\ldots,w_n,0}^{(k)}$ and omit the superscript when $k = 1$, i.e. the cuspidal representation with these Hodge weights is unique up to equivalence.

6.5 Some related problems

In this subsection we explain some representation-theoretic problems motivated by our conjectural classification of discrete global Arthur parameters for $\mathbf{F}_4$.

6.5.1 Theta correspondence between PGL_2 and $\mathbf{F}_4$

Inside an exceptional group $\mathbf{E}_{7,3}$ of Lie type E_7 and $\mathbb{Q}$-rank 3, which is split over every finite prime p, there is a reductive dual pair $\mathrm{PGL}_2 \times \mathbf{F}_4$, so we have an exceptional theta correspondence between representations of PGL_2 and $\mathbf{F}_4$.

For a level one cuspidal automorphic representation $\pi \in \Pi_{\mathrm{alg}}^{\perp}(\mathrm{PGL}_2)$, by Savin's work on this exceptional theta correspondence [Sav94], if the theta lift $\Theta(\pi)$ of π to $\mathbf{F}_4$ is nonzero, then its corresponding discrete global Arthur parameter is $\psi = \pi[6] \oplus [9] \oplus [5]$. By Proposition 6.3.6, we see that $\mathrm{m}(\pi_\psi)$ is always 1, admitting Arthur's conjectures. This predicts that the global theta lift $\Theta(\pi)$ is nonzero for any $\pi \in \Pi_{\mathrm{alg}}^{\perp}(\mathrm{PGL}_2)$, and we will prove this result in another paper in progress.

Remark 6.5.1. For $\pi \in \Pi_{\mathrm{alg}}^{\perp}(\mathrm{PGL}_2)$, the archimedean theta lift of π_∞ is isomorphic to the irreducible representation $\mathrm{V}_{n\varpi_4}$ of F_4 for some n. For readers interested in this exceptional theta correspondence, we list in Table 7 the dimensions of $\mathrm{V}_{n\varpi_4}^{\mathcal{F}_{4,\mathrm{I}}(\mathbb{Z})}$ and $\mathrm{V}_{n\varpi_4}^{\mathcal{F}_{4,\mathrm{E}}(\mathbb{Z})}$ for $n \leq 40$.

6.5.2 Theta correspondence between $\mathbf{G}_2$ and $\mathbf{F}_4$

Inside an exceptional group $\mathbf{E}_{8,4}$ of Lie type E_8 and $\mathbb{Q}$-rank 4, there is a reductive dual pair $\mathbf{G}_2 \times \mathbf{F}_4$, where $\mathbf{G}_2$ is the generic fiber of the split Chevalley group of Lie type G_2.

In [Dal23], Dalal classifies level one quaternionic discrete automorphic representations of $\mathbf{G}_2$. The exceptional theta correspondence from $\mathbf{G}_2$ to $\mathbf{F}_4$ is functorial, so for a level one quaternionic discrete automorphic representation of $\mathbf{G}_2$, if its global theta lift to $\mathbf{F}_4$ is nonzero, then we can describe the corresponding discrete global Arthur parameters in $\Psi_{\mathrm{AJ}}(\mathbf{F}_4)$. The discrete global Arthur parameters of $\mathbf{F}_4$ involving in this correspondence are:

- $\mathrm{Sym}^2\,\pi[3] \oplus \pi[4] \oplus \pi[2] \oplus [5]$, $\pi \in \Pi_{\mathrm{alg}}^{\perp}(\mathrm{PGL}_2)$,

- $\mathrm{Sym}^2\,\pi_1[3] \oplus (\pi_1 \otimes \pi_2[3]) \oplus [5]$, where $\pi_1, \pi_2 \in \Pi_{\mathrm{alg}}^{\perp}(\mathrm{PGL}_2)$ satisfy $\mathrm{w}(\pi_2) = 3\mathrm{w}(\pi_1) + 2$,
- and $\pi[3] \oplus [5]$, where $\pi \in \Pi_{\mathrm{alg}}^{\mathrm{G}_2}(\mathrm{PGL}_7)$.

According to Proposition 6.3.7, Proposition 6.3.10 and Proposition 6.3.13, for every ψ among these discrete global Arthur parameters, we have $\mathrm{m}(\pi_\psi) = 1$. This predicts that the global theta lift of any level one quaternionic discrete automorphic representation of $\mathbf{G}_2$ to $\mathbf{F}_4$ is nonzero, which is proved by Pollack in [Pol23, §8].

Remark 6.5.2. For any quaternionic discrete series π of $\mathbf{G}_2(\mathbb{R})$, the archimedean theta lift of π is isomorphic to the irreducible representation $\mathrm{V}_{n\varpi_3}$ of F_4 for some n. For readers interested in this exceptional theta correspondence, we list in Table 8 the dimensions of $\mathrm{V}_{n\varpi_3}^{\mathcal{F}_{4,\mathrm{I}}(\mathbb{Z})}$ and $\mathrm{V}_{n\varpi_3}^{\mathcal{F}_{4,\mathrm{E}}(\mathbb{Z})}$ for $n \leq 30$.

A Figures and tables

$$
\begin{pmatrix}
4 & 2 & 2 & -3 & -1 & -1 & -1 & -3 & -3 & -3 & -2 & -2 & -2 & -2 & -2 & -4 & -4 & -4 & -4 & -2 & -2 & -2 & -2 & -4 & -4 & -4 & -4 \\
2 & 4 & 2 & -2 & -2 & -2 & -2 & -4 & -4 & -4 & -4 & -3 & -1 & -1 & -1 & -3 & -3 & -3 & -2 & -2 & -2 & -2 & -2 & -4 & -4 & -4 & -4 \\
2 & 2 & 4 & -2 & -2 & -2 & -2 & -4 & -4 & -4 & -4 & -2 & -2 & -2 & -2 & -4 & -4 & -4 & -4 & -3 & -1 & -1 & -1 & -3 & -3 & -3 & -2 \\
-3 & -2 & -2 & 5 & 1 & 1 & 1 & 4 & 4 & 4 & 2 & 2 & 2 & 2 & 2 & 4 & 4 & 4 & 4 & 2 & 2 & 2 & 2 & 4 & 4 & 4 & 4 \\
-1 & -2 & -2 & 1 & 5 & 1 & 1 & 4 & 4 & 4 & 4 & 2 & 0 & 2 & 2 & 2 & 2 & 2 & 3 & 2 & 0 & 0 & 0 & 2 & 2 & 2 & 0 \\
-1 & -2 & -2 & 1 & 1 & 5 & 1 & 4 & 2 & 2 & 4 & 2 & 0 & 0 & 2 & 2 & 2 & 2 & 1 & 2 & 2 & 0 & 0 & 2 & 3 & 3 & 2 \\
-1 & -2 & -2 & 1 & 1 & 1 & 5 & 2 & 4 & 2 & 4 & 2 & 0 & 0 & 0 & 2 & 2 & 2 & 1 & 2 & 2 & 2 & 0 & 3 & 2 & 3 & 2 \\
-3 & -4 & -4 & 4 & 4 & 4 & 2 & 8 & 6 & 6 & 6 & 4 & 2 & 2 & 3 & 5 & 5 & 6 & 5 & 4 & 2 & 2 & 2 & 5 & 6 & 5 & 4 \\
-3 & -4 & -4 & 4 & 4 & 2 & 4 & 6 & 8 & 6 & 6 & 4 & 2 & 3 & 2 & 6 & 5 & 5 & 5 & 4 & 2 & 2 & 2 & 5 & 5 & 6 & 4 \\
-3 & -4 & -4 & 4 & 4 & 2 & 2 & 6 & 6 & 8 & 6 & 4 & 2 & 3 & 3 & 5 & 6 & 5 & 5 & 4 & 2 & 2 & 2 & 6 & 5 & 5 & 4 \\
-2 & -4 & -4 & 2 & 4 & 4 & 4 & 6 & 6 & 6 & 8 & 4 & 0 & 2 & 2 & 4 & 4 & 4 & 3 & 4 & 3 & 1 & 1 & 5 & 5 & 5 & 3 \\
-2 & -3 & -2 & 2 & 2 & 2 & 2 & 4 & 4 & 4 & 4 & 5 & 1 & 1 & 1 & 4 & 4 & 4 & 2 & 2 & 2 & 2 & 2 & 4 & 4 & 4 & 4 \\
-2 & -1 & -2 & 2 & 0 & 0 & 0 & 2 & 2 & 2 & 0 & 1 & 5 & 1 & 1 & 4 & 4 & 4 & 4 & 2 & 0 & 2 & 2 & 2 & 2 & 2 & 3 \\
-2 & -1 & -2 & 2 & 2 & 0 & 0 & 2 & 3 & 3 & 2 & 1 & 1 & 5 & 1 & 4 & 2 & 2 & 4 & 2 & 0 & 0 & 2 & 2 & 2 & 2 & 1 \\
-2 & -1 & -2 & 2 & 2 & 2 & 0 & 3 & 2 & 3 & 2 & 1 & 1 & 1 & 5 & 2 & 4 & 2 & 4 & 2 & 0 & 0 & 0 & 2 & 2 & 2 & 1 \\
-4 & -3 & -4 & 4 & 2 & 2 & 2 & 5 & 6 & 5 & 4 & 4 & 4 & 4 & 2 & 8 & 6 & 6 & 6 & 4 & 2 & 2 & 3 & 5 & 5 & 6 & 5 \\
-4 & -3 & -4 & 4 & 2 & 2 & 2 & 5 & 5 & 6 & 4 & 4 & 4 & 2 & 4 & 6 & 8 & 6 & 6 & 4 & 2 & 3 & 2 & 6 & 5 & 5 & 5 \\
-4 & -3 & -4 & 4 & 2 & 2 & 2 & 6 & 5 & 5 & 4 & 4 & 4 & 2 & 2 & 6 & 6 & 8 & 6 & 4 & 2 & 3 & 3 & 5 & 6 & 5 & 5 \\
-4 & -2 & -4 & 4 & 3 & 1 & 1 & 5 & 5 & 5 & 3 & 2 & 4 & 4 & 4 & 6 & 6 & 6 & 8 & 4 & 0 & 2 & 2 & 4 & 4 & 4 & 3 \\
-2 & -2 & -3 & 2 & 2 & 2 & 2 & 4 & 4 & 4 & 4 & 2 & 2 & 2 & 2 & 4 & 4 & 4 & 4 & 5 & 1 & 1 & 1 & 4 & 4 & 4 & 2 \\
-2 & -2 & -1 & 2 & 0 & 2 & 2 & 2 & 2 & 2 & 3 & 2 & 0 & 0 & 0 & 2 & 2 & 2 & 0 & 1 & 5 & 1 & 1 & 4 & 4 & 4 & 4 \\
-2 & -2 & -1 & 2 & 0 & 0 & 2 & 2 & 2 & 2 & 1 & 2 & 2 & 0 & 0 & 2 & 3 & 3 & 2 & 1 & 1 & 5 & 1 & 4 & 2 & 2 & 4 \\
-2 & -2 & -1 & 2 & 0 & 0 & 0 & 2 & 2 & 2 & 1 & 2 & 2 & 2 & 0 & 3 & 2 & 3 & 2 & 1 & 1 & 1 & 5 & 2 & 4 & 2 & 4 \\
-4 & -4 & -3 & 4 & 2 & 2 & 3 & 5 & 5 & 6 & 5 & 4 & 2 & 2 & 2 & 5 & 6 & 5 & 4 & 4 & 4 & 4 & 2 & 8 & 6 & 6 & 6 \\
-4 & -4 & -3 & 4 & 2 & 3 & 2 & 6 & 5 & 5 & 5 & 4 & 2 & 2 & 2 & 5 & 5 & 6 & 4 & 4 & 4 & 2 & 4 & 6 & 8 & 6 & 6 \\
-4 & -4 & -3 & 4 & 2 & 3 & 3 & 5 & 6 & 5 & 5 & 4 & 2 & 2 & 2 & 6 & 5 & 5 & 4 & 4 & 4 & 2 & 2 & 6 & 6 & 8 & 6 \\
-4 & -4 & -2 & 4 & 0 & 2 & 2 & 4 & 4 & 4 & 3 & 4 & 3 & 1 & 1 & 5 & 5 & 5 & 3 & 2 & 4 & 4 & 4 & 6 & 6 & 6 & 8
\end{pmatrix}
$$

Figure 1: The gram matrix of $(\mathrm{J}_{\mathbb{Z}}, \langle\ ,\ \rangle_{\mathrm{E}})$ in the basis $\mathcal{B}$ given in (2.6)

$$\sigma_1 = \begin{pmatrix}
1 & 2 & 1 & 0 & -1 & 0 & -2 & -1 & -2 & -1 & -2 & -1 & 0 & -1 & 0 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -2 & -2 & -1 & -1 \\
1 & 1 & 1 & -1 & 0 & -1 & -1 & -1 & -1 & -1 & -1 & 0 & 0 & -1 & -1 & -1 & -1 & 0 & -1 & -1 & -1 & -1 & 0 & -2 & -1 & -1 & -1 \\
1 & 2 & 1 & -1 & 0 & -2 & -1 & -2 & -1 & -1 & -2 & -1 & 0 & 0 & 0 & -1 & -1 & -1 & 0 & 0 & -2 & -1 & -1 & -2 & -2 & -1 & -2 \\
-1 & -2 & -1 & 1 & 0 & 2 & 2 & 2 & 1 & 1 & 2 & 1 & 0 & 0 & 1 & 1 & 2 & 1 & 1 & 1 & 2 & 2 & 0 & 3 & 2 & 1 & 2 \\
-1 & -1 & -1 & 1 & -1 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 2 & 1 & 1 & 1 & 2 & 1 & 1 & 2 & 2 & 1 & 2 \\
0 & 0 & -1 & 0 & -1 & 1 & 1 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & -1 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & -1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\
1 & 1 & 1 & -1 & 1 & -2 & -1 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & -1 & -1 & -1 & -1 & -1 & -2 & -1 & 0 & -2 & -2 & -2 & -2 \\
0 & 1 & 1 & -1 & 0 & -1 & -1 & -2 & -1 & -1 & -1 & -1 & -1 & 0 & -1 & -1 & -2 & -1 & -1 & -1 & 0 & -1 & -1 & -1 & -1 & 0 & -1 \\
1 & 1 & 0 & 0 & -1 & -1 & 0 & -1 & 0 & -1 & -1 & -1 & 1 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & -1 & -1 & 0 & -2 & -1 & 0 & -1 \\
0 & 0 & 1 & -1 & 1 & 0 & -1 & 0 & -1 & -1 & -1 & 1 & -1 & -1 & 1 & -1 & 0 & 0 & 0 & -1 & -1 & 1 & -1 & 0 & -1 & -1 & 0 \\
-1 & -3 & -1 & 1 & 0 & 2 & 3 & 2 & 3 & 1 & 3 & 2 & 0 & 1 & -1 & 2 & 1 & 1 & 0 & 1 & 2 & 2 & 2 & 3 & 3 & 2 & 3 \\
-1 & -2 & -1 & 1 & 2 & 2 & 2 & 3 & 3 & 1 & 3 & 2 & 0 & 1 & 0 & 2 & 1 & 2 & 1 & 1 & 2 & 1 & 1 & 2 & 3 & 2 & 2 \\
-1 & -1 & 0 & 2 & 2 & 0 & 0 & 2 & 2 & 2 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 2 & 1 & 1 \\
0 & 0 & 1 & 0 & 1 & 0 & -1 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
1 & 3 & 1 & -1 & -2 & -2 & -2 & -3 & -3 & -3 & -4 & -2 & 1 & -1 & 0 & -1 & -1 & -1 & 0 & -1 & -2 & -1 & -1 & -3 & -3 & -2 & -2 \\
0 & 1 & 0 & 0 & 0 & -1 & -1 & 0 & -1 & 1 & -1 & 0 & 0 & -1 & 1 & -1 & 1 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & -1 & -1 & -1 \\
0 & 1 & 1 & 0 & 1 & -1 & -1 & -1 & -1 & 0 & -1 & -1 & -1 & 0 & 1 & -1 & -1 & -1 & 0 & -1 & 0 & -1 & -1 & -1 & -1 & 0 & -1 \\
1 & 0 & -1 & -1 & -2 & 1 & 1 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & -1 & 1 & -1 & 0 & 0 & 0 & -1 & 0 & 0 \\
-1 & -2 & 0 & 1 & -1 & 0 & 2 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 2 & 2 & 1 & 3 & 2 & 1 & 2 \\
0 & -1 & 0 & 0 & -1 & 0 & 1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & -1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 1 \\
1 & 0 & 1 & -1 & -1 & 0 & -1 & -1 & -1 & -2 & -2 & 0 & 0 & -2 & -1 & -1 & -1 & -1 & -2 & -1 & 0 & 0 & -1 & -1 & -1 & 0 & 0 \\
1 & 0 & 0 & -2 & -1 & 1 & 0 & -1 & -1 & -1 & 0 & 0 & -1 & -1 & -1 & -1 & -1 & -1 & -2 & 0 & 0 & -1 & -1 & -1 & -1 & 0 & -1 \\
0 & 1 & 0 & -1 & 1 & 1 & -1 & 0 & -1 & 0 & 1 & 0 & -1 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & -2 & 0 & -1 & 0 & 0 & -1 \\
0 & 1 & 0 & 1 & 0 & 0 & -1 & 1 & 0 & -1 & -1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & 0 \\
1 & 0 & 0 & -1 & 1 & 1 & -1 & 1 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & -1 & -1 & -1 & -1 & 0 & -1 \\
-1 & 0 & 0 & 2 & 0 & -2 & 0 & 0 & 0 & 2 & 0 & -1 & 1 & 1 & 1 & 0 & 1 & 1 & 2 & 0 & 0 & 1 & 2 & 1 & 1 & -1 & 1
\end{pmatrix}$$

$$\sigma_2 = \begin{pmatrix}
2 & 2 & 1 & -2 & 0 & -1 & -1 & -2 & -2 & -1 & -1 & -1 & -1 & -1 & -1 & -2 & -2 & -2 & -2 & -1 & -1 & -2 & -2 & -2 & -3 & -2 & -3 \\
2 & 2 & 2 & -1 & -1 & 0 & -2 & -2 & -3 & -2 & -2 & -1 & -2 & -2 & 0 & -3 & -2 & -3 & -3 & -2 & 0 & -2 & -2 & -2 & -2 & -2 & -2 \\
1 & 1 & 1 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & -1 & 0 & 0 & -1 & 0 & -1 & 0 & -1 & -1 & 0 & -1 & 0 & -1 & 0 & -1 & -1 & -1 \\
-1 & -1 & -1 & 0 & 0 & -1 & 2 & 0 & 2 & -1 & 0 & -1 & 1 & 1 & 0 & 1 & 0 & 1 & 2 & 0 & -1 & 1 & 1 & -1 & 0 & 0 & 0 \\
-1 & 0 & -1 & 0 & 1 & 0 & 2 & 0 & 2 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 2 & 1 & 0 & 0 & -1 & 0 & 0 & 1 & -1 \\
0 & -1 & 0 & 0 & 2 & 1 & 1 & 1 & 2 & 1 & 2 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & -1 & 1 & 1 & 2 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & -1 & 0 & -1 & -1 & 0 & 1 & 1 & -2 & 1 & -1 & 1 & 0 \\
1 & 1 & 1 & -1 & -2 & 1 & -1 & -1 & -3 & -1 & -1 & 0 & 0 & -2 & 0 & -1 & 0 & -1 & -2 & 0 & 0 & 0 & -1 & 0 & -1 & -1 & 0 \\
0 & 0 & 0 & 1 & 1 & 1 & -1 & 2 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 2 & 0 & 1 \\
0 & 0 & 1 & 1 & 0 & 1 & -1 & 0 & 0 & 1 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 1 & 0 & -1 & 1 & 0 & 2 & 1 \\
1 & 1 & 0 & -1 & -1 & -2 & -2 & -1 & -2 & -2 & -3 & -2 & 1 & 0 & 0 & -1 & -1 & -1 & 0 & -2 & -3 & -1 & 1 & -3 & -2 & -4 & -2 \\
-1 & -2 & 0 & 1 & -1 & -1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & -1 & 1 & 0 & 1 & 1 & 0 & 1 & 2 & 3 & 1 & 2 & 1 & 3 \\
0 & 0 & 1 & 0 & 0 & -2 & -1 & -1 & -1 & -1 & -2 & -1 & 0 & 0 & 0 & -1 & -1 & -1 & 0 & -2 & -1 & 1 & 1 & -1 & -1 & -2 & 0 \\
0 & 1 & 1 & -1 & 0 & -1 & 0 & -1 & -2 & -1 & -1 & -1 & -1 & -1 & 0 & -2 & -1 & -1 & -1 & -1 & 0 & 1 & -1 & 0 & -1 & -2 & -1 \\
-1 & 1 & 0 & 1 & 0 & -1 & -1 & 0 & -1 & 0 & -1 & -1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 1 & 0 & 0 & -1 & 0 \\
0 & 1 & -1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & -1 & -1 & -1 & 0 & -1 & 1 & -1 \\
1 & 1 & 0 & -2 & 0 & 1 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & -2 & 0 & -1 & 0 & 0 & -1 & 1 & -1 & -1 & -2 & -1 & -1 & -1 & -2 \\
0 & 1 & 0 & 0 & 2 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & -1 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & -1 & -2 \\
1 & -2 & 0 & 0 & -1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & -2 & 1 & 0 & 0 & -1 & 1 & 1 & 0 & 1 & 1 & 1 & 2 & 2 \\
-3 & -2 & -2 & 2 & -1 & 1 & 3 & 2 & 3 & 1 & 2 & 2 & 2 & 2 & 0 & 4 & 3 & 4 & 3 & 2 & 2 & 3 & 3 & 3 & 4 & 3 & 4 \\
-2 & -1 & -1 & 1 & -1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 2 & 2 & 0 & 3 & 2 & 2 & 2 & 2 & 1 & 1 & 3 & 2 & 3 & 2 & 3 \\
-1 & 1 & 0 & 0 & -1 & -1 & -2 & -1 & -1 & -1 & -2 & -1 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & -1 & -1 & 1 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 1 & -1 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & -2 & 1 & -1 & 1 & 1 & 0 \\
2 & 0 & 1 & -1 & 2 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -2 & -1 & 0 & -2 & -2 & -2 & -2 & -1 & 0 & -2 & -1 & -1 & -1 & -1 & -2 \\
1 & 1 & 1 & 0 & 0 & -2 & -1 & -2 & -1 & 0 & -2 & -1 & -1 & -1 & 0 & -2 & -1 & -2 & -1 & -2 & -1 & 0 & -2 & -1 & -3 & -1 & -1 \\
2 & 2 & 2 & -3 & -1 & -1 & -3 & -3 & -4 & -3 & -3 & -2 & -1 & -2 & -1 & -3 & -3 & -3 & -3 & -2 & -2 & -3 & -1 & -4 & -3 & -3 & -3 \\
0 & 0 & -1 & 1 & 0 & 1 & 3 & 1 & 1 & 1 & 2 & 1 & -1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 2 & 2 & -2 & 2 & 0 & 0 & 0
\end{pmatrix}$$

Figure 2: Generators σ_1 and σ_2 of $\mathcal{F}_{4,\mathrm{E}}(\mathbb{Z})$ as 27×27 matrices in the basis $\mathcal{B}$ of $\mathrm{J}_{\mathbb{Z}}$

s	o(c_s)	i(c_s)	s	o(c_s)	i(c_s)	s	o(c_s)	i(c_s)
(1,0,0,0,0)	1	(27,351,2925,52)	(2,1,1,0,1)	9	(3,3,0,1)	(4,4,2,0,1)	20	(4,3,-8,0)
(0,0,0,0,1)	2	(-5,-1,45,20)	(0,1,0,1,2)	10	(-2,1,0,6)	(7,0,1,1,3)	20	(4,9,16,4)
(0,1,0,0,0)	2	(3,-9,-35,-4)	(0,2,0,1,1)	10	(0,-1,0,0)	(2,1,3,1,2)	21	(0,0,2,0)
(0,0,1,0,0)	3	(0,0,9,-2)	(4,2,0,0,1)	10	(10,49,160,10)	(4,2,1,2,1)	21	(2,1,-1,0)
(1,0,0,0,1)	3	(0,0,9,7)	(0,0,0,1,4)	12	(-4,0,21,15)	(0,4,0,1,6)	24	(-2,0,3,7)
(1,1,0,0,0)	3	(9,36,90,7)	(0,1,0,2,1)	12	(-1,2,-2,1)	(0,6,0,1,4)	24	(0,-2,-1,1)
(0,0,0,1,0)	4	(-1,3,-3,0)	(0,2,0,1,2)	12	(-1,0,0,3)	(1,2,3,2,1)	24	(0,0,3,-1)
(0,1,0,0,1)	4	(-1,-1,1,4)	(0,4,0,1,0)	12	(2,-6,-15,-3)	(2,4,2,1,2)	24	(1,-2,-2,-1)
(1,0,1,0,0)	4	(3,3,1,0)	(1,0,3,0,1)	12	(0,0,5,-1)	(3,1,3,1,3)	24	(0,0,1,1)
(2,0,0,0,1)	4	(7,27,77,8)	(1,1,1,1,1)	12	(0,0,1,0)	(3,5,1,1,2)	24	(2,-2,-7,-1)
(2,1,0,0,0)	4	(15,111,545,20)	(1,3,1,0,1)	12	(2,-4,-11,-2)	(4,0,2,1,5)	24	(-1,0,2,5)
(1,1,0,0,1)	5	(2,1,0,2)	(1,4,1,0,0)	12	(3,-6,-26,-3)	(4,2,2,1,3)	24	(1,0,0,1)
(0,0,0,1,1)	6	(-2,2,-3,5)	(2,0,0,1,3)	12	(-2,0,5,8)	(4,2,4,1,0)	24	(2,0,-1,-1)
(0,1,0,0,2)	6	(-3,0,10,11)	(2,0,2,1,0)	12	(1,0,2,-1)	(6,2,0,3,1)	24	(3,4,2,1)
(0,1,0,1,0)	6	(0,0,1,-1)	(2,1,0,1,2)	12	(0,0,1,3)	(6,2,4,0,1)	24	(4,6,3,1)
(0,2,0,0,1)	6	(1,-4,-6,-1)	(2,2,0,1,1)	12	(2,0,-3,0)	(7,2,1,1,3)	24	(4,8,11,3)
(1,0,1,0,1)	6	(0,0,1,2)	(2,4,0,0,1)	12	(4,0,-19,-1)	(2,4,2,1,4)	28	(0,-1,0,1)
(1,1,1,0,0)	6	(3,0,-8,-1)	(3,0,1,1,1)	12	(2,2,1,1)	(3,4,1,3,1)	28	(1,-1,-1,-1)
(2,0,0,1,0)	6	(4,8,9,2)	(3,3,1,0,0)	12	(6,12,5,2)	(2,4,6,0,1)	30	(1,-2,1,-2)
(2,1,0,0,1)	6	(6,18,37,5)	(4,0,2,0,1)	12	(5,12,18,3)	(3,6,1,1,4)	30	(1,-2,-3,0)
(3,0,1,0,0)	6	(12,72,289,14)	(4,1,0,0,3)	12	(3,6,14,5)	(6,1,0,5,1)	30	(1,1,0,0)
(4,0,0,0,1)	6	(16,128,681,23)	(5,0,1,1,0)	12	(8,32,85,7)	(6,4,2,2,1)	30	(3,2,-3,0)
(4,1,0,0,0)	6	(21,216,1450,35)	(6,1,0,0,2)	12	(11,62,238,13)	(8,0,2,1,6)	30	(1,1,4,4)
(1,0,0,1,1)	7	(-1,1,-1,3)	(2,1,1,1,1)	13	(1,0,0,0)	(12,1,0,3,2)	30	(7,25,60,6)
(2,1,1,0,0)	7	(6,15,20,3)	(2,2,2,0,1)	14	(2,-1,-4,-1)	(1,4,3,4,1)	36	(0,0,2,-1)
(0,0,0,1,2)	8	(-3,1,5,10)	(4,1,0,1,2)	14	(3,5,7,3)	(2,8,2,1,4)	36	(1,-3,-4,-1)
(0,1,0,1,1)	8	(-1,1,-1,2)	(1,0,2,1,2)	15	(-1,1,0,2)	(4,6,2,1,7)	40	(0,-1,0,2)
(0,2,0,1,0)	8	(1,-3,-3,-2)	(4,2,1,1,0)	15	(5,10,9,2)	(8,2,6,1,3)	40	(2,1,0,0)
(1,1,1,0,1)	8	(1,-1,-1,0)	(1,1,3,1,1)	18	(0,0,4,-1)	(1,6,5,1,5)	42	(0,-1,1,0)
(1,2,1,0,0)	8	(3,-3,-17,-2)	(2,2,2,1,1)	18	(1,-1,0,-1)	(10,2,4,1,6)	42	(2,2,2,2)
(2,0,0,1,1)	8	(1,1,1,2)	(4,1,0,1,4)	18	(0,0,4,5)	(1,12,7,2,3)	60	(1,-3,-2,-2)
(2,2,0,0,1)	8	(5,9,5,2)	(6,2,2,0,1)	18	(7,23,48,5)	(6,4,6,1,12)	60	(-1,0,1,4)
(3,1,1,0,0)	8	(9,39,111,8)	(2,4,2,1,0)	20	(2,-3,-8,-2)	(10,2,10,1,6)	60	(1,0,1,0)
(1,1,0,1,1)	9	(0,0,0,1)	(3,0,1,3,1)	20	(0,1,0,0)	(11,12,1,3,5)	60	(3,1,-6,0)

Table 4: Kac coordinates, Orders and invariants i (defined in §3.5) of the rational torsion conjugacy classes of F_4

s	$n_1(s)$	$n_2(s)$	s	$n_1(s)$	$n_2(s)$
(1,0,0,0,0)	1	1	(1,1,1,1,1)	435456000	105670656
(0,0,0,0,1)	723	819	(1,3,1,0,1)	101606400	0
(0,1,0,0,0)	459900	68796	(2,0,0,1,3)	1612800	0
(0,0,1,0,0)	6540800	2283008	(2,0,2,1,0)	24192000	13208832
(1,0,0,0,1)	121920	139776	(2,1,0,1,2)	43545600	0
(1,1,0,0,0)	268800	34944	(2,2,0,1,1)	14515200	17611776
(0,0,0,1,0)	249480	137592	(2,4,0,0,1)	4112640	0
(0,1,0,0,1)	2835000	0	(3,0,1,1,1)	7257600	0
(1,0,1,0,0)	14968800	3302208	(3,3,1,0,0)	4838400	0
(2,0,0,0,1)	23400	58968	(4,0,2,0,1)	14515200	4402944
(2,1,0,0,0)	37800	0	(5,0,1,1,0)	3628800	0
(1,1,0,0,1)	1741824	0	(2,1,1,1,1)	0	48771072
(0,0,0,1,1)	497280	0	(2,2,2,0,1)	223948800	11321856
(0,1,0,1,0)	44150400	8805888	(4,2,1,1,0)	34836480	0
(0,2,0,0,1)	10483200	2201472	(1,1,3,1,1)	232243200	0
(1,0,1,0,1)	74995200	17611776	(2,2,2,1,1)	154828800	105670656
(1,1,1,0,0)	67737600	8805888	(6,2,2,0,1)	19353600	0
(2,0,0,1,0)	1881600	2935296	(2,4,2,1,0)	87091200	0
(2,1,0,0,1)	604800	0	(4,4,2,0,1)	52254720	0
(3,0,1,0,0)	806400	0	(2,1,3,1,2)	199065600	30191616
(4,0,0,0,1)	6720	0	(4,2,1,2,1)	0	60383232
(1,0,0,1,1)	0	4313088	(0,4,0,1,6)	7257600	0
(2,1,1,0,0)	24883200	539136	(0,6,0,1,4)	21772800	0
(0,0,0,1,2)	272160	0	(1,2,3,2,1)	174182400	0
(0,1,0,1,1)	10886400	0	(2,4,2,1,2)	174182400	52835328
(0,2,0,1,0)	22680000	6604416	(3,1,3,1,3)	261273600	0
(1,1,1,0,1)	342921600	0	(3,5,1,1,2)	87091200	0
(1,2,1,0,0)	32659200	0	(4,2,2,1,3)	58060800	52835328
(2,0,0,1,1)	5443200	6604416	(4,2,4,1,0)	65318400	0
(2,2,0,0,1)	5715360	0	(6,2,4,0,1)	50803200	0
(3,1,1,0,0)	5443200	0	(2,4,2,1,4)	149299200	22643712
(1,1,0,1,1)	77414400	0	(2,4,6,0,1)	34836480	0
(2,1,1,0,1)	19353600	35223552	(6,4,2,2,1)	139345920	0
(0,2,0,1,1)	38320128	0	(2,8,2,1,4)	116121600	0
(4,2,0,0,1)	1741824	0	(4,6,2,1,7)	104509440	0
(0,2,0,1,2)	29030400	8805888	(8,2,6,1,3)	104509440	0
(0,4,0,1,0)	10886400	0	(6,4,6,1,12)	69672960	0
(1,0,3,0,1)	47174400	0			

Table 5: Kac coordinates of the conjugacy classes of F_4 whose intersections with $\mathcal{F}_{4,\mathrm{I}}(\mathbb{Z})$ and $\mathcal{F}_{4,\mathrm{E}}(\mathbb{Z})$ are not both empty

λ	$d(\lambda)$	λ	$d(\lambda)$	λ	$d(\lambda)$	λ	$d(\lambda)$	λ	$d(\lambda)$
(0,0,0,2)	1	(0,0,1,9)	7	(0,1,1,7)	7	(0,0,0,13)	8	(2,0,4,1)	13
(0,0,0,3)	1	(0,0,2,7)	6	(0,1,2,5)	9	(0,0,1,11)	15	(2,1,0,6)	16
(0,0,0,4)	1	(0,0,3,5)	6	(0,1,3,3)	14	(0,0,2,9)	20	(2,1,1,4)	17
(0,0,2,0)	1	(0,0,4,3)	4	(0,1,4,1)	4	(0,0,3,7)	27	(2,1,2,2)	25
(0,0,0,5)	1	(0,0,5,1)	1	(0,2,0,6)	11	(0,0,4,5)	34	(2,1,3,0)	8
(0,0,1,3)	1	(0,1,0,8)	2	(0,2,1,4)	9	(0,0,5,3)	30	(2,2,0,3)	4
(0,0,0,6)	3	(0,1,1,6)	3	(0,2,2,2)	15	(0,0,6,1)	14	(2,2,1,1)	9
(0,0,2,2)	1	(0,1,2,4)	4	(0,2,3,0)	2	(0,1,0,10)	11	(2,3,0,0)	6
(0,0,0,7)	1	(0,1,3,2)	3	(0,3,0,3)	3	(0,1,1,8)	23	(3,0,0,7)	1
(0,0,1,5)	1	(0,1,4,0)	1	(0,3,1,1)	3	(0,1,2,6)	39	(3,0,1,5)	9
(0,0,2,3)	1	(0,2,0,5)	1	(0,4,0,0)	6	(0,1,3,4)	44	(3,0,2,3)	7
(0,0,0,8)	4	(0,2,1,3)	3	(1,0,0,10)	3	(0,1,4,2)	37	(3,0,3,1)	8
(0,0,1,6)	1	(0,2,2,1)	1	(1,0,1,8)	7	(0,1,5,0)	13	(3,1,0,4)	12
(0,0,2,4)	1	(0,3,0,2)	2	(1,0,2,6)	10	(0,2,0,7)	11	(3,1,1,2)	7
(0,0,4,0)	2	(1,0,0,9)	1	(1,0,3,4)	11	(0,2,1,5)	32	(3,1,2,0)	8
(0,0,0,9)	4	(1,0,1,7)	3	(1,0,4,2)	8	(0,2,2,3)	36	(4,0,0,5)	2
(0,0,1,7)	2	(1,0,2,5)	2	(1,0,5,0)	4	(0,2,3,1)	26	(4,0,1,3)	3
(0,0,2,5)	1	(1,0,3,3)	3	(1,1,0,7)	2	(0,3,0,4)	21	(4,0,2,1)	2
(0,0,3,3)	2	(1,0,4,1)	1	(1,1,1,5)	9	(0,3,1,2)	21	(4,1,0,2)	4
(0,1,3,0)	1	(1,1,0,6)	3	(1,1,2,3)	8	(0,3,2,0)	14	(4,1,1,0)	1
(0,3,0,0)	1	(1,1,1,4)	2	(1,1,3,1)	9	(0,4,0,1)	5	(5,0,1,1)	1
(1,1,0,4)	1	(1,1,2,2)	4	(1,2,0,4)	8	(1,0,0,11)	3	(5,1,0,0)	3
(3,1,0,0)	1	(1,2,1,1)	2	(1,2,1,2)	5	(1,0,1,9)	13		
(0,0,0,10)	5	(1,3,0,0)	1	(1,2,2,0)	5	(1,0,2,7)	20		
(0,0,1,8)	4	(2,0,0,7)	1	(1,3,0,1)	1	(1,0,3,5)	32		
(0,0,2,6)	6	(2,0,1,5)	2	(2,0,0,8)	5	(1,0,4,3)	26		
(0,0,4,2)	3	(2,0,2,3)	1	(2,0,1,6)	4	(1,0,5,1)	21		
(0,0,5,0)	1	(2,0,3,1)	1	(2,0,2,4)	10	(1,1,0,8)	18		
(0,1,1,5)	1	(2,1,0,4)	2	(2,0,3,2)	4	(1,1,1,6)	27		
(0,1,3,1)	1	(2,1,1,2)	1	(2,0,4,0)	5	(1,1,2,4)	46		
(0,2,0,4)	1	(2,1,2,0)	1	(2,1,1,3)	5	(1,1,3,2)	31		
(0,2,2,0)	1	(3,0,1,3)	1	(2,1,2,1)	2	(1,1,4,0)	20		
(1,0,0,8)	1	(3,1,0,2)	1	(2,2,0,2)	8	(1,2,0,5)	10		
(1,0,1,6)	1	(0,0,0,12)	13	(3,0,0,6)	4	(1,2,1,3)	28		
(1,0,2,4)	1	(0,0,1,10)	6	(3,0,1,4)	3	(1,2,2,1)	16		
(1,0,3,2)	1	(0,0,2,8)	15	(3,0,2,2)	3	(1,3,0,2)	18		
(1,2,0,2)	1	(0,0,3,6)	15	(3,0,3,0)	2	(1,3,1,0)	2		
(2,0,0,6)	2	(0,0,4,4)	15	(3,2,0,0)	2	(2,0,0,9)	4		
(2,0,2,2)	1	(0,0,5,2)	4	(4,0,0,4)	3	(2,0,1,7)	12		
(2,2,0,0)	1	(0,0,6,0)	11	(4,0,2,0)	2	(2,0,2,5)	16		
(0,0,0,11)	5	(0,1,0,9)	2	(6,0,0,0)	3	(2,0,3,3)	21		

Table 6: The nonzero $d(\lambda)$ for $\lambda = (\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ such that $2\lambda_1 + 3\lambda_2 + 2\lambda_3 + \lambda_4 \leq 13$

n	$d_1(n)$	$d_2(n)$	n	$d_1(n)$	$d_2(n)$	n	$d_1(n)$	$d_2(n)$	n	$d_1(n)$	$d_2(n)$
1	0	0	11	4	1	21	83	209	31	4112	24425
2	1	0	12	8	5	22	130	413	32	6294	38234
3	1	0	13	6	2	23	169	590	33	8904	54760
4	1	0	14	12	8	24	280	1138	34	13284	82989
5	1	0	15	13	8	25	368	1629	35	18664	117447
6	2	1	16	20	18	26	601	2915	36	27332	173760
7	1	0	17	22	22	27	835	4253	37	38024	242971
8	3	1	18	37	58	28	1323	7161	38	54627	351485
9	3	1	19	39	63	29	1868	10455	39	75354	486013
10	4	1	20	67	150	30	2919	16962	40	106332	689219

Table 7: Dimensions $d_1(n) = \dim \mathrm{V}_{n\varpi_4}^{\mathcal{F}_{4,\mathrm{I}}(\mathbb{Z})}$ and $d_2(n) = \dim \mathrm{V}_{n\varpi_4}^{\mathcal{F}_{4,\mathrm{E}}(\mathbb{Z})}$ for $n \leq 40$

n	$d_1(n)$	$d_2(n)$	n	$d_1(n)$	$d_2(n)$
1	0	0	16	699558	4607562
2	1	0	17	1899450	12528178
3	0	0	18	4951537	32636950
4	1	1	19	12298529	81088431
5	0	1	20	29444006	194120684
6	4	7	21	67821302	447181025
7	2	14	22	151304284	997568542
8	32	136	23	326873722	2155210696
9	84	583	24	686811782	4528418428
10	497	2936	25	1404333622	9259307898
11	1765	11764	26	2802604042	18478677233
12	7111	46299	27	5463354204	36021961176
13	24173	159701	28	10425639768	68740584631
14	80166	526081	29	19491910968	128517811865
15	241776	1594526	30	35762551274	235797459916

Table 8: Dimensions $d_1(n) = \dim \mathrm{V}_{n\varpi_3}^{\mathcal{F}_{4,\mathrm{I}}(\mathbb{Z})}$ and $d_2(n) = \dim \mathrm{V}_{n\varpi_3}^{\mathcal{F}_{4,\mathrm{E}}(\mathbb{Z})}$ for $n \leq 30$

$\mathrm{w}(\lambda)$	λ	$\dim \mathcal{A}_{\mathrm{V}_\lambda}(\mathbf{F}_4)$	$\Psi_\lambda(\mathbf{F}_4)$
16	(0,0,0,0)	2	$[9] \oplus [17]$
			$\Delta_{11}[6] \oplus [5] \oplus [9]$
20	(0,0,0,2)	1	$\Delta_{15}[6] \oplus [5] \oplus [9]$
22	(0,0,0,3)	1	$\Delta_{17}[6] \oplus [5] \oplus [9]$
24	(0,0,0,4)	1	$\Delta_{19}[6] \oplus [5] \oplus [9]$
	(0,0,2,0)	1	$\mathrm{Sym}^2 \Delta_{11}[3] \oplus \Delta_{11}[4] \oplus \Delta_{11}[2] \oplus [5]$
26	(0,0,0,5)	1	$\Delta_{21}[6] \oplus [5] \oplus [9]$
	(0,0,1,3)	1	$\Delta_{24,16,8,0}[3] \oplus [5]$
28	(0,0,0,6)	3	$\Delta_{23}^{(2)}[6] \oplus [5] \oplus [9]$
			$\Delta_{26,20,6,0}[3] \oplus [5]$
	(0,0,2,2)	1	$\Delta_{26,16,10,0}[3] \oplus [5]$
30	(0,0,0,7)	1	$\Delta_{25}[6] \oplus [5] \oplus [9]$
	(0,0,1,5)	1	$\Delta_{28,20,8,0}[3] \oplus [5]$
	(0,0,2,3)	1	$\Delta_{28,18,10,0}[3] \oplus [5]$
32	(0,0,0,8)	4	$\Delta_{27}^{(2)}[6] \oplus [5] \oplus [9]$
			$\Delta_{30,24,6,0}^{(2)}[3] \oplus [5]$
	(0,0,1,6)	1	$\Delta_{30,22,8,0}[3] \oplus [5]$
	(0,0,2,4)	1	$\Delta_{30,20,10,0}[3] \oplus [5]$
	(0,0,4,0)	2	$\mathrm{Sym}^2 \Delta_{15}[3] \oplus \Delta_{15}[4] \oplus \Delta_{15}[2] \oplus [5]$
			$\Delta_{30,16,14,0}[3] \oplus [5]$
34	(0,0,0,9)	4	$\Delta_{29}^{(2)}[6] \oplus [5] \oplus [9]$
			$\Delta_{32,26,6,0}^{(2)}[3] \oplus [5]$
	(0,0,1,7)	2	$\Delta_{32,24,8,0}^{(2)}[3] \oplus [5]$
	(0,0,2,5)	1	$\Delta_{32,22,10,0}[3] \oplus [5]$
	(0,0,3,3)	2	$\Delta_{32,20,12,0}^{(2)}[3] \oplus [5]$
	(0,1,3,0)	1	$\Delta_{32,16,14,6,0} \oplus \mathrm{Spin}\, \Delta_{32,16,14,6,0} \oplus [1]$
	(0,3,0,0)	1	$\mathrm{Sym}^3 \Delta_{11}[2] \oplus \mathrm{Sym}^2 \Delta_{11}[3] \oplus \Delta_{11}[4] \oplus [1]$
	(1,1,0,4)	1	$\Delta_{30,20,10,8,0} \oplus \mathrm{Spin}\, \Delta_{30,20,10,8,0} \oplus [1]$
	(3,1,0,0)	1	$\wedge^* \Delta_{19,7} \oplus (\Delta_{19,7} \otimes \Delta_{15}) \oplus \Delta_{19,7}[2] \oplus \Delta_{15}[2] \oplus [1]$

Table 9: Elements of nonempty $\Psi_\lambda(\mathbf{F}_4)$ for the weights λ such that $\mathrm{w}(\lambda) \leq 34$

λ	$\dim \mathcal{A}_{\mathrm{V}_\lambda}(\mathbf{F}_4)$	$\Psi_\lambda(\mathbf{F}_4)$
(0,0,0,10)	5	$\Delta_{31}^{(2)}[6] \oplus [5] \oplus [9]$
		$\Delta_{34,28,6,0}^{(3)}[3] \oplus [5]$
(0,0,1,8)	4	$\wedge^*\Delta_{21,13} \oplus (\Delta_{21,13} \otimes \Delta_{15}) \oplus (\Delta_{21,13} \otimes \Delta_{11}) \oplus (\Delta_{15} \otimes \Delta_{11}) \oplus [1]$
		$\Delta_{34,26,8,0}^{(3)}[3] \oplus [5]$
(0,0,2,6)	6	$\Delta_{34,24,10,0}^{(5)}[3] \oplus [5]$
		$\Delta_{34,24,10,4,0} \oplus \mathrm{Spin}\, \Delta_{34,24,10,4,0} \oplus [1]$
(0,0,4,2)	3	$\Delta_{34,20,14,0}^{(2)}[3] \oplus [5]$
		$\Delta_{34,20,14,4,0} \oplus \mathrm{Spin}\, \Delta_{32,20,14,4,0} \oplus [1]$
(0,0,5,0)	1	$\mathrm{Sym}^2\, \Delta_{17}[3] \oplus \Delta_{17}[4] \oplus \Delta_{17}[2] \oplus [5]$
(0,1,1,5)	1	$\Delta_{34,22,10,6,0} \oplus \mathrm{Spin}\, \Delta_{34,22,10,6,0} \oplus [1]$
(0,1,3,1)	1	$\Delta_{34,18,14,6,0} \oplus \mathrm{Spin}\, \Delta_{34,18,14,6,0} \oplus [1]$
(0,2,0,4)	1	$\Delta_{34,20,10,8,0} \oplus \mathrm{Spin}\, \Delta_{32,16,14,6,0} \oplus [1]$
(0,2,2,0)	1	$\wedge^*\Delta_{21,13} \oplus (\Delta_{21,13} \otimes \Delta_{15}) \oplus \Delta_{21,13}[2] \oplus \Delta_{15}[2] \oplus [1]$
(1,0,0,8)	1	$\Delta_{32,26,8,6,0} \oplus \mathrm{Spin}\, \Delta_{32,26,8,6,0} \oplus [1]$
(1,0,1,6)	1	$\Delta_{32,24,10,6,0} \oplus \mathrm{Spin}\, \Delta_{32,24,10,6,0} \oplus [1]$
(1,0,2,4)	1	$\Delta_{32,22,12,6,0} \oplus \mathrm{Spin}\, \Delta_{32,22,12,6,0} \oplus [1]$
(1,0,3,2)	1	$\Delta_{32,20,14,6,0} \oplus \mathrm{Spin}\, \Delta_{32,20,14,6,0} \oplus [1]$
(1,2,0,2)	1	ψ_0
(2,0,0,6)	2	$\Delta_{30,24,10,8,0}^{(2)} \oplus \mathrm{Spin}\, \Delta_{30,24,10,8,0} \oplus [1]$
(2,0,2,2)	1	$\Delta_{30,20,14,8,0} \oplus \mathrm{Spin}\, \Delta_{30,20,14,8,0} \oplus [1]$
(2,2,0,0)	1	$\wedge^*\Delta_{21,9} \oplus (\Delta_{21,9} \otimes \Delta_{15}) \oplus \Delta_{21,9}[2] \oplus \Delta_{15}[2] \oplus [1]$

Table 10: Elements of nonempty $\Psi_\lambda(\mathbf{F}_4)$ for the weights λ such that $\mathrm{w}(\lambda) = 36$

λ	$F_4(\lambda)$	λ	$F_4(\lambda)$	λ	$F_4(\lambda)$	λ	$F_4(\lambda)$	λ	$F_4(\lambda)$
(1,2,0,2)	1	(1,2,2,0)	5	(1,1,3,2)	22	(0,1,3,5)	70	(2,0,2,6)	28
(0,1,2,4)	2	(2,0,2,4)	2	(1,1,4,0)	11	(0,1,4,3)	68	(2,0,3,4)	32
(0,1,4,0)	1	(2,0,3,2)	2	(1,2,0,5)	7	(0,1,5,1)	49	(2,0,4,2)	35
(0,2,1,3)	2	(2,1,1,3)	3	(1,2,1,3)	22	(0,2,0,8)	31	(2,0,5,0)	12
(0,3,0,2)	2	(2,1,2,1)	2	(1,2,2,1)	13	(0,2,1,6)	61	(2,1,0,7)	10
(1,0,3,3)	1	(2,2,0,2)	4	(1,3,0,2)	12	(0,2,2,4)	92	(2,1,1,5)	42
(1,1,1,4)	1	(3,0,0,6)	1	(1,3,1,0)	2	(0,2,3,2)	74	(2,1,2,3)	46
(1,1,2,2)	2	(3,0,2,2)	2	(2,0,1,7)	2	(0,2,4,0)	35	(2,1,3,1)	41
(1,2,1,1)	2	(3,2,0,0)	1	(2,0,2,5)	3	(0,3,0,5)	26	(2,2,0,4)	39
(2,1,0,4)	2	(0,0,3,7)	3	(2,0,3,3)	9	(0,3,1,3)	61	(2,2,1,2)	34
(2,1,2,0)	1	(0,0,4,5)	6	(2,0,4,1)	5	(0,3,2,1)	40	(2,2,2,0)	24
(0,0,3,6)	1	(0,0,5,3)	8	(2,1,0,6)	11	(0,4,0,2)	28	(2,3,0,1)	2
(0,0,4,4)	1	(0,0,6,1)	4	(2,1,1,4)	9	(0,4,1,0)	8	(3,0,0,8)	5
(0,0,5,2)	1	(0,1,0,10)	2	(2,1,2,2)	21	(1,0,0,12)	1	(3,0,1,6)	6
(0,0,6,0)	1	(0,1,1,8)	6	(2,1,3,0)	2	(1,0,1,10)	4	(3,0,2,4)	21
(0,1,1,7)	1	(0,1,2,6)	19	(2,2,0,3)	1	(1,0,2,8)	23	(3,0,3,2)	13
(0,1,2,5)	3	(0,1,3,4)	18	(2,2,1,1)	8	(1,0,3,6)	36	(3,0,4,0)	14
(0,1,3,3)	6	(0,1,4,2)	25	(2,3,0,0)	4	(1,0,4,4)	50	(3,1,0,5)	2
(0,1,4,1)	2	(0,1,5,0)	4	(3,0,1,5)	2	(1,0,5,2)	34	(3,1,1,3)	21
(0,2,0,6)	4	(0,2,0,7)	2	(3,0,2,3)	2	(1,0,6,0)	24	(3,1,2,1)	13
(0,2,1,4)	4	(0,2,1,5)	20	(3,0,3,1)	3	(1,1,0,9)	6	(3,2,0,2)	20
(0,2,2,2)	8	(0,2,2,3)	21	(3,1,0,4)	4	(1,1,1,7)	50	(3,2,1,0)	2
(0,2,3,0)	2	(0,2,3,1)	19	(3,1,1,2)	5	(1,1,2,5)	69	(4,0,0,6)	2
(0,3,0,3)	3	(0,3,0,4)	19	(3,1,2,0)	3	(1,1,3,3)	86	(4,0,1,4)	3
(0,3,1,1)	2	(0,3,1,2)	10	(4,1,0,2)	3	(1,1,4,1)	57	(4,0,2,2)	7
(0,4,0,0)	1	(0,3,2,0)	13	(0,0,2,10)	4	(1,2,0,6)	56	(4,0,3,0)	1
(1,0,2,6)	2	(0,4,0,1)	2	(0,0,3,8)	13	(1,2,1,4)	72	(4,1,1,1)	6
(1,0,3,4)	2	(1,0,2,7)	4	(0,0,4,6)	27	(1,2,2,2)	93	(4,2,0,0)	1
(1,0,4,2)	4	(1,0,3,5)	11	(0,0,5,4)	26	(1,2,3,0)	17	(5,0,0,4)	2
(1,1,1,5)	4	(1,0,4,3)	9	(0,0,6,2)	24	(1,3,0,3)	18	(5,0,2,0)	2
(1,1,2,3)	4	(1,0,5,1)	11	(0,0,7,0)	8	(1,3,1,1)	34	(7,0,0,0)	1
(1,1,3,1)	6	(1,1,0,8)	7	(0,1,0,11)	1	(1,4,0,0)	9		
(1,2,0,4)	7	(1,1,1,6)	15	(0,1,1,9)	21	(2,0,0,10)	3		
(1,2,1,2)	3	(1,1,2,4)	27	(0,1,2,7)	44	(2,0,1,8)	9		

Table 11: The nonzero $F_4(\lambda)$ for the weights λ such that $w(\lambda) \leq 44$

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